MATH 1300 D01 Assignment #4

Due: Thursday, December 22^{th}, 2016

Instructions:

SHOW YOUR WORK to get full marks.

All assignments must be handed in on UMLearn as **one** PDF file. Late assignments will not be accepted. Failure to follow the instructions will result in a mark of 0.

This assignment covers topics from Unit 6 and Unit 7.

The total number of marks for this assignment is 80.

Again for this assignment, I will use angled brackets for (row) vectors to distinguish them from points. For example $\mathbf{u} = \langle 3, 1, 2 \rangle$ is a vector but $P = (3, 1, 2)$ is a point. This convention is not used in the notes, and may not be used for the final exam.

1. (15 points) For each of the following transformations, determine if T is linear. If T is linear, find the standard matrix for T and determine is T is one to one and/or onto.

   (a) $T : \mathbb{R}^3 \to \mathbb{R}^3$ where $T(\mathbf{v}) = \mathbf{v} \times \langle 1, 2, 3 \rangle$

   (b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ where $T(\mathbf{v}) = T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} v_1 + 1 \\ v_2 + 2 \\ v_3 + 3 \end{bmatrix}$

   (c) $T : \mathbb{R}^3 \to \mathbb{R}^3$ where $T(\mathbf{v}) = T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} v_2 + v_3 \\ v_1 + v_3 \\ v_1 + v_2 \end{bmatrix}$

   (d) $T : \mathbb{R}^3 \to \mathbb{R}^3$ where $T(\mathbf{v}) = T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} \mathbf{v} \cdot \mathbf{w} \\ \mathbf{v} \cdot (2\mathbf{w}) \\ \mathbf{v} \cdot (3\mathbf{w}) \end{bmatrix}$ where $\mathbf{w} = \langle 1, 2, 3 \rangle$

   (e) $T : \mathbb{R}^3 \to \mathbb{R}^3$ where $T(\mathbf{v}) = T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} \mathbf{v} \cdot \mathbf{v} \\ \mathbf{v} \cdot \mathbf{w} \\ 0 \end{bmatrix}$ where $\mathbf{w} = \langle 1, 2, 3 \rangle$

2. (12 points) Find the standard matrix for the following transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$.

   (a) Rotation by an angle of $\frac{2\pi}{3}$ (standard rotations are counterclockwise).

   (b) Reflection about the line $y = \sqrt{2}x$.

   (c) Dilation by a factor of $\frac{1}{3}$.

   (d) Rotation by $-\frac{\pi}{4}$ followed by reflection about the line $y = x$.

   (e) Rotation by $\frac{\pi}{2}$ followed by reflection about the x-axis.

3. (28 points) For each of the following matrics A, find all the eigenvalues and associated eigenspaces.

   (a) $A = \begin{pmatrix} 1 & 2 \\ 5 & -2 \end{pmatrix}$

   (b) $A = \begin{pmatrix} 11 & -32 \\ 4 & -13 \end{pmatrix}$

   (c) $A = \begin{pmatrix} 0 & 2 & -1 \\ 2 & -3 & 2 \\ -1 & 2 & 0 \end{pmatrix}$

   (d) $A = \begin{pmatrix} 2 & 1 & -1 \\ 2 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix}$

4. (12 points) Suppose T is a linear transformation. Suppose you know the following about T:

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 10 \\ 7 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 6 \\ 4 \end{bmatrix},$$

   (a) Find the standard matrix for T.

   (b) Find all eigenvalues of T.

   (c) Is $[T]$ diagonalizable? (Justify your answer.)

   (d) Is T invertible? (You do not necessarily need to find the inverse.)

5. (8 points) The past performance of the matches Blue team plays against Green team leads to the following predictive results. If Blue team wins a match, the probability that they will win the next match is 80%. On the other hand, if Blue team loses a match, the probability they will lose the next match is 70%. If Green team wins today, determine the probability Blue team will win a match a long time from now.

6. (5 points) Suppose A is a square matrix that satisfies the matrix equation $A^3 = A$. Find, with justification, all possible eigenvalues of A.