MATH 1300 – D01 – Assignment 2

Solutions

Answer the following questions **on separate sheets** and scan the result to produce a single PDF file. Please show your work. Unclear or not fully justified answers **will not** get full marks.

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1. Let
$$A = \begin{pmatrix} 2 & 3 \\ 0 & -2 \end{pmatrix}.$$

1.a. Compute $P(\lambda) = \det(A - \lambda\mathbb{I})$, where $\mathbb{I}$ is the appropriate identity matrix. [5]

Solution. We have

$$\begin{aligned} A - \lambda\mathbb{I} &= \begin{pmatrix} 2 & 3 \\ 0 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 \\ 0 & -2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} 2-\lambda & 3 \\ 0 & -2-\lambda \end{pmatrix}. \end{aligned}$$

As this matrix is an upper triangular matrix, its determinant is the product of the diagonal entries, so

$$P(\lambda) = (2-\lambda)(-2-\lambda) = -(2-\lambda)(2+\lambda) = \lambda^2 - 4.$$

1.b. Compute $P(A)$, where P is the polynomial obtained in **1.a.** in which constant terms have been replaced by "constant times identity" (e.g., if $P(\lambda) = a\lambda^2 + b\lambda + c$, then $P(A) = aA^2 + bA + c\mathbb{I}$). [5]

Solution. We proceed as explained: we write that for a matrix M,

$$P(M) = M^2 - 4\mathbb{I}.$$

To apply this to A, we need A^2. We have

$$A^2 = \begin{pmatrix} 2 & 3 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}.$$

So, finally,

$$P(A) = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

2. Assuming the matrices have sizes such that all operations are valid, simplify [5]

$$(AC^{-1})^{-1}(AC^{-1})(AC^{-1})^{-1}AD^{-1}.$$

Solution. Recall that if M, N are invertible matrices of the same size, then $(MN)^{-1} = N^{-1}M^{-1}$. Recall also that $(M^{-1})^{-1} = M$ for any invertible matrix. Applying this gives

$$\begin{aligned}(AC^{-1})^{-1}(AC^{-1})(AC^{-1})^{-1}AD^{-1} &= CA^{-1}AC^{-1}CA^{-1}AD^{-1} \\ &= CD^{-1}.\end{aligned}$$

Note: I believe my intent was for the last D^{-1} to be a C^{-1}, which would have made the whole expression simplify to $\mathbb{I}$. Too bad..

3. Prove that if A is an invertible matrix and B is row equivalent to A, then B is also invertible. [5]

4. Let A be a 3×3 matrix with entries equal to 0 or 1. Find the range of $\det(A)$, i.e, the largest a and smallest b such that $\det(A) \in [a, b]$ for all matrices A of the prescribed form. [5]

Solution. The easiest is probably to simply enumerate the possible cases. Bear in mind that whenever two rows or columns are equal, the determinant is zero. The determinant is also zero if there is a row or column of zeros. The determinant of a matrix equals that of its transpose, so we need only reason one way, the other is obtained by transposing.

Any matrix with less than three 1's has necessarily at least a row of zeros and thus has zero determinant.

The most simple 3×3 matrix with entries in $\{0, 1\}$ and with a nonzero determinant is the identity matrix

$$\mathbb{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which has determinant 1. Any other matrix with a single 1 per row is either obtained by permutation of rows of $\mathbb{I}_3$ and so has determinant -1 (in case of 1 permutation) or 1 (2 permutations). So this far, we have a range of $[-1, 1]$.

Consider the following matrices:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

From these matrices, it is clear that having eight 1's is also not possible.