Student number: ______________________________
Surname: ______________________________
First name: ______________________________

This assignment consists of THREE pages and covers topics from unit 1 of the course. Please make sure that you answer and submit all pages.

1. State whether the following matrices are in row-echelon form (REF), reduced row-echelon form (RREF), both or neither.

$$(a)\quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad (b)\quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad (c)\quad \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \qquad (d)\quad \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution: (a) Neither, (b) Both, (c) REF, (d) Both

2. Solve the system of linear equations using elimination:

$$\begin{aligned} x - y &= 2 \\ 2x - 3y &= 1 \end{aligned}$$

Solution: Multiply both sides of the first equation by 2 yields

$$\begin{aligned} 2x - 2y &= 4 \\ 2x - 3y &= 1 \end{aligned}$$

Taking the second equation and subtracting the first leads to

$$\begin{aligned} 2x - 2y &= 4 \\ -y &= -3 \end{aligned}$$

Dividing the second equation by -1 yields

$$\begin{aligned} 2x - 2y &= 4 \\ y &= 3 \end{aligned}$$

Then since $y = 3$ we get $2x - 2y = 4 \Rightarrow x = 5$. Thus $(x, y) = (5, 3)$ is the solution.

3. Solve the following systems of linear equations.

$$\begin{aligned} x_1 + 4x_2 - 2x_3 + 3x_5 &= 0 \\ 2x_1 + 8x_2 - 5x_3 - 2x_4 + 6x_5 - 3x_6 &= -1 \\ 5x_3 + 10x_4 + 15x_6 &= 5 \end{aligned}$$

Solution:

$$\left[\begin{array}{cccccc|c} 1 & 4 & -2 & 0 & 3 & 0 & 0 \\ 2 & 8 & -5 & -2 & 6 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccccc|c} 1 & 4 & -2 & 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \end{array}\right] \xrightarrow{R_2 \to -R_2}$$

$$\left[\begin{array}{cccccc|c} 1 & 4 & -2 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \end{array}\right] \xrightarrow[R_3 \to R_3 - 5R_2]{R_1 \to R_1 + 2R_2} \left[\begin{array}{cccccc|c} 1 & 4 & 0 & 4 & 3 & 6 & 2 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The matrix is now in RREF. Let $x_2 = r$, $x_4 = s$, $x_5 = t$, $x_6 = u$, then the first row can be written as

$$x_1 + 4x_2 + 4x_4 + 3x_5 + 6x_6 = 2 \quad \Rightarrow x_1 = 2 - 4r - 4s - 3t - 6u.$$

From the second row

$$x_3 + 2x_4 + 3x_6 = 1 \quad \Rightarrow \quad x_3 = 1 - 2s - 3u.$$

Thus the solution is

$$(2 - 4r - 4s - 3t - 6u, r, 1 - 2s - 3u, s, t, u).$$

4. Find the equation of a parabola that passes through the points $(2, \frac{1}{2})$, $(-1, 2)$ and $(0, -\frac{1}{2})$. Do this by first assuming the parabola has the form $y = ax^2 + bx + c$ then write out the linear system of equations that you need to solve. Write this as an augmented matrix and use Gauss-Jordan elimination to solve for a, b and c.

Solution: Using $x = 2$, $y = 0$ we get the equation $\frac{1}{2} = a(2)^2 + b(2) + c = 4a + 2b + c$. Using $x = -1$, $y = 2$ we get the equation $2 = a(-1)^2 + b(-1) + c = a - b + c$. Using $x = 0$, $y = -1$ we get the equation $-\frac{1}{2} = a(0)^2 + b(0) + c = c$. Hence we get the system

$$\begin{aligned} 4a + 2b + c &= \frac{1}{2} \\ a - b + c &= 2 \\ c &= -\frac{1}{2} \end{aligned}$$

Using augmented matrices to solve yields,

$$\left[\begin{array}{ccc|c} 4 & 2 & 1 & \frac{1}{2} \\ 1 & -1 & 1 & 2 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 4 & 2 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right] \xrightarrow{R_2 \to R_2 - 4R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 6 & -3 & -\frac{15}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right] \xrightarrow{R_2 \to \frac{1}{6}R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -\frac{1}{2} & -\frac{5}{4} \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{3}{4} \\ 0 & 1 & -\frac{1}{2} & -\frac{5}{4} \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right] \xrightarrow[R_2 \to R_2 + \frac{1}{2}R_3]{R_1 \to R_1 - \frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right]$$

Hence $a = 1$, $b = -\frac{3}{2}$, $c = \frac{1}{2}$ and the parabola has equation $y = x^2 - \frac{3}{2}x - \frac{1}{2}$