Student number: ______________________

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This assignment consists of FOUR pages. Show your solutions.

1. Let A, B and C be the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 & 2 \\ 4 & 0 & 4 \\ 2 & 1 & 0 \end{bmatrix}.$$

Compute the following matrices or state why the matrix is not defined.

(a) $A^T A + C$

Solution:

$$A^T A + C = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 2 \\ 4 & 0 & 4 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 8 & 6 \\ 8 & 8 & 8 \\ 6 & 8 & 10 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 2 \\ 4 & 0 & 4 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 9 & 8 \\ 12 & 8 & 12 \\ 8 & 9 & 10 \end{bmatrix}$$

(b) $CA^T + BA$

Solution: Not defined since CA^T is 3×2 and BA is 2×3.

(c) $\frac{1}{4}C(BA)^T$

Solution:

$$\frac{1}{4}C(BA)^T = \frac{1}{4}C\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}\right)^T = \frac{1}{4}C\left(\begin{bmatrix} 7 & 6 & 5 \\ 15 & 14 & 13 \end{bmatrix}\right)^T$$

$$= \frac{1}{4}\begin{bmatrix} 0 & 1 & 2 \\ 4 & 0 & 4 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 7 & 15 \\ 6 & 14 \\ 5 & 13 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 12 & 28 \\ 5 & 11 \end{bmatrix}$$

(d) $2B^{-1}A + AC$

Solution:

$$B^{-1} = \frac{1}{4-6}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$$2B^{-1}A + AC = 2\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 4 & 0 & 4 \\ 2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -4 & -10 \\ 0 & 4 & 8 \end{bmatrix} + \begin{bmatrix} 14 & 4 & 10 \\ 10 & 4 & 14 \end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 \\ 10 & 8 & 22 \end{bmatrix}$$

2. Consider the following matrix,

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}.$$

(a) Compute M^{-1} by augmenting M with a 3×3 identity matrix and using row reduction.

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_2]{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -2 & 4 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to -\frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & -\frac{1}{2} \end{array}\right] \xrightarrow[R_2 \to R_2 - 2R_3]{R_1 \to R_1 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & -2 & 1 & -\frac{1}{2} \end{array}\right]$$

Thus the inverse is $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 2 & -1 & 1 \\ -2 & 1 & -\frac{1}{2} \end{bmatrix}$.

(b) Write M and M^{-1} as the product of elementary matrices.

Solution: The row operations in part (a) can be written as $E_6E_5E_4E_3E_2E_1M = I$ where E_1 represents the first row operation ($R_2 \to R_2 - 2R_1$), E_2 represents the second row operation ($R_1 \to R_1 - R_2$), and so on:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

The inverse matrices are also elementary matrices and represent the reverse operations. Example: E_1^{-1} represents ($R_2 \to R_2 + 2R_1$).

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix},$$

$$E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad E_5^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_6^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that since E_4 represents the row operation $R_3 \to -\frac{1}{2}R_3$ (divide by -2), E_4^{-1} represents $R_3 \to -2R_3$ (multiply by -2). Do not forget the negative sign.
Since $E_6E_5E_4E_3E_2E_1M = I$ then

$$M^{-1} = E_6E_5E_4E_3E_2E_1, \qquad M = (E_6E_5E_4E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}.$$

(c) Calculate $\det(M)$ by cofactor expansion.

Solution:

$$\det(M) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{vmatrix} = (1)(-1)^{1+1}\begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix} + (2)(-1)^{2+1}\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} + (0)(-1)^{3+1}\begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix}$$

$$= -2 + 0 + 0 = -2$$

(d) Calculate $\det(M)$ by using row reduction to a triangular matrix (see unit 3.4).

Solution:

$$\det(M) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{vmatrix} \overset{R_2 \to R_2 - 2R_1}{=} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 2 \end{vmatrix} \overset{\substack{R_1 \to R_1 - R_2 \\ R_3 \to R_3 - 2R_2}}{=} \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{vmatrix} = -2$$

Note that we use vertical lines instead of square brackets when referring to determinants of the matrix. If you used square brackets in the calculation above you may get deductions in the exam due to improper notation. That is because notation matters! See below.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix}, \qquad \text{These two matrices are not equal.}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 2 \end{vmatrix}, \qquad \text{These two determinants are equal.}$$

3. Consider the following system of equations.

$$\begin{aligned} x + 2y &= a \\ a^2x + 2y &= 1 \end{aligned}$$

(a) How many solutions does this system have if $a = -1$?

Solution: If $a = -1$ we obtain the following augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & -1 \\ 1 & 2 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 0 & 2 \end{array}\right]$$

This is inconsistent (the last line can be written as $0x + 0y = 2$ which is not possible). There is no solution.

(b) How many solutions does this system have if $a = 1$?

Solution: If $a = 1$ we obtain the following augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 2 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

Let $y = t \in \mathbb{R}$. This has infinitely many solutions of the form $(x, xy) = (1 - 2t, t)$.

(c) How many solutions does this system have if $a \neq 1$ and $a \neq -1$?

Solution: Here are two ways to solve this. The second solution is shorter.
Solution 1: If $a \neq 1$ and $a \neq -1$ we obtain the following augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & a \\ a^2 & 2 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - a^2 R_1} \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & 2 - 2a^2 & 1 - a^3 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{2-2a^2} R_2} \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & \frac{1-a^3}{2-2a^2} \end{array}\right] \xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{cc|c} 1 & 0 & a - 2\left(\frac{1-a^3}{2-2a^2}\right) \\ 0 & 1 & \frac{1-a^3}{2-2a^2} \end{array}\right]$$

This has one unique solution $x = a - 2\left(\frac{1-a^3}{2-2a^2}\right)$ (which simplifies to $x = -\frac{1}{a+1}$) and $y = \frac{1-a^3}{2-2a^2}$ as long as $a \neq 1$ and $a \neq -1$.

Solution 2: Using the determinant for 2×2 matrices, $\begin{vmatrix} 1 & 2 \\ a^2 & 2 \end{vmatrix} = 2 - 2a^2$ which is non-zero as long as $a \neq 1$ and $a \neq -1$. Thus in this case there is always a unique solution to the system. If you also want to solve this system, you can use Cramer's rule:

$$x = \frac{\begin{vmatrix} a & 2 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ a^2 & 2 \end{vmatrix}} = \frac{2a - 2}{2 - 2a^2} = -\frac{1}{a+1}, \quad y = \frac{\begin{vmatrix} 1 & a \\ a^2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ a^2 & 2 \end{vmatrix}} = \frac{1 - a^3}{2 - 2a^2}$$