Student number: ____________________

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This assignment consists of TWO pages. Show your solutions.

1. Let $\vec{u} = (1, 0, 3)$, $\vec{v} = (2, -1, 0)$ and $\vec{w} = (2, 0, -2)$. Calculate the following:

(a) $(\vec{u} \times \vec{v}) \times \vec{w}$

Solution:

$$(\vec{u} \times \vec{v}) \times \vec{w} = \big((1, 0, 3) \times (2, -1, 0)\big) \times (2, 0, -2) = (3, 6, -1) \times (2, 0, -2) = (-12, 4, -12)$$

(b) $(\vec{w} \times \vec{v}) + \vec{u}$

Solution:

$$(\vec{w} \times \vec{v}) + \vec{u} = \big((2, 0, -2) \times (2, -1, 0)\big) + (1, 0, 3) = (-2, -4, -2) + (1, 0, 3) = (-1, -4, 1)$$

(c) $\vec{u} \cdot \vec{w} + \|\vec{v}\|$

Solution:

$$(1, 0, 3) \cdot (2, 0, -2) = 2 + 0 + (-6) = -4, \qquad |\vec{v}| = \sqrt{2^2 + (-1)^2 + 0^2} = \sqrt{5}$$

$$\vec{u} \cdot \vec{w} + \|\vec{v}\| = -4 + \sqrt{5}$$

(d) $\text{proj}_{\vec{v}}\vec{u}$ (projection of $\vec{u}$ onto $\vec{v}$)

Solution: As before, $|\vec{v}| = \sqrt{2^2 + (-1)^2 + 0^2} = \sqrt{5}$.

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|v\|^2}\vec{v} = \frac{(1, 0, 3) \cdot (2, -1, 0)}{(\sqrt{5})^2}(2, -1, 0) = \frac{2}{5}(2, -1, 0) = \Big(\frac{4}{5}, -\frac{2}{5}, 0\Big).$$

(e) The volume of the parallelepiped determined by the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$.

Solution: Compute the following scalar triple product. Here I used cofactor expansion about the second column.

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 1 & 0 & 3 \\ 2 & -1 & 0 \\ 2 & 0 & -2 \end{vmatrix} = -(0)\begin{vmatrix} 2 & 0 \\ 2 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 3 \\ 2 & -2 \end{vmatrix} - (0)\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = 8$$

The volume of the parallelpiped is $|\vec{u} \cdot (\vec{v} \times \vec{w})| = |8| = 8$.

2. Find the parametric equations of the line of intersection of the planes $x + 2y + z = 1$ and $x + y + 5z = 3$.

Solution: We are solving for x, y and z that satisfies both equations of planes. Hence we need to solve that system of equations.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 1 & 1 & 5 & 3 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & 4 & 2 \end{array}\right] \xrightarrow{R_2 \leftrightarrow -R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & -4 & -2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 5 \\ 0 & 1 & -4 & -2 \end{array}\right]$$

Let $z = t$. The equations above yield $x = 5 - 9t$ and $y = -2 + 4t$. The parametric equations of the line is $(x, y, z) = (5 - 9t, -2 + 4t, t)$.

Note that there are other solutions and parametric forms possible. For example, $(x, y, z) = (5 + 9t, -2 - 4t, -t)$ is also a correct answer.

3. Find the distance between the point $P(1, 2, 3)$ and the plane $x + 2y + 2z = 0$.

Solution: Let Q be a point on the plane. For example, we can choose $Q = (0, 0, 0)$ since this satisfies the equations of the plane. This, $\vec{PQ} = (0-1, 0-2, 0-3) = (-1, -2, -3)$. Furthermore, the normal vector to the plane is $\vec{n} = (1, 2, 2)$. This has norm $\|\vec{n}\| = \sqrt{1^2 + 2^2 +^2} = \sqrt{9} = 3$. The distance of the point to the plane is the magnitude of the projection of $\vec{PQ}$ onto $\vec{n}$:

$$\left|\text{proj}_{\vec{n}} \vec{PQ}\right| = \left|\frac{\vec{PQ} \cdot \vec{n}}{\|\vec{n}\|^2} \vec{n}\right| = \frac{|\vec{PQ} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|(-1, -2, -3) \cdot (1, 2, 2)|}{3} = \frac{11}{3}.$$

An alternative solution is to use the formula from unit 5.4.

$$\text{distance} = \frac{|ax_0 + bx_0 + cz_0 + d|}{\sqrt{a^2 + b^c + c^2}} = \frac{|(1)(1) + (2)(2) + (2)(3)|}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{11}{3}.$$