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This assignment consists of THREE pages. Show your solutions.

1. Find the standard matrix corresponding to the following linear transformations. Evaluate all trigonometric expressions.

(a) $T(x, y, z) = (x - z, x + y - \frac{z}{2}, 4y - 3z)$

Solution:

$$T(\vec{e}_1) = (1, 1, 0), \quad T(\vec{e}_2) = (0, 1, 4), \quad T(\vec{e}_3) = (-1, -\frac{1}{2}, -3)$$

The standard matrix is

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -\frac{1}{2} \\ 0 & 4 & -3 \end{bmatrix}.$$

(b) Rotation on $\mathbb{R}^2$ by $\frac{\pi}{3}$ clockwise

Solution:

$$T(\vec{e}_1) = (\cos(\theta), \sin(\theta)), \quad T(\vec{e}_2) = (-\sin(\theta), \cos(\theta)).$$

The standard matrix can be found by substituting $\theta = -\frac{\pi}{3}$,

$$\begin{bmatrix} \cos(-\frac{\pi}{3}) & -\sin(-\frac{\pi}{3}) \\ \sin(-\frac{\pi}{3}) & \cos(-\frac{\pi}{3}) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

(c) Reflection on $\mathbb{R}^2$ about the y-axis

Solution:

$$T(\vec{e}_1) = (-1, 0), \quad T(\vec{e}_2) = (0, 1).$$

The standard matrix is

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

(d) Reflection on $\mathbb{R}^2$ about the line $y = x$

Solution:

$$T(\vec{e}_1) = (0, 1), \quad T(\vec{e}_2) = (1, 0).$$

The standard matrix is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

2. Consider a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$T(\vec{e}_1 - \vec{e}_2) = (-1, -2), \qquad T(3\vec{e}_2) = (0, 6).$$

Use the properties of linear transformations to find the following:

(a) Find $T(\vec{e}_2)$.

Solution: Since $T(2\vec{e}_2) = 3T(\vec{e}_2)$, then $T(\vec{e}_2) = \frac{1}{3}(0, 6) = (0, 2)$.

(b) Find $T(\vec{e}_1)$.

Solution:

$$T(\vec{e}_1) = T(\vec{e}_1 - \vec{e}_2 + \vec{e}_2) = T(\vec{e}_1 - \vec{e}_2) + T(\vec{e}_2) = (-1, -2) + (0, 2) = (-1, 0).$$

(c) Find the standard matrix associated with T.

Solution:

$$\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

(d) Evaluate $T\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$.

Solution:

$$T\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(e) Find the standard matrix associated with T^{-1}.

Solution:

$$\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \frac{1}{-2-0} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

(f) Apply T^{-1} to your answer in part (d) and verify that you get $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ back.

Solution:

$$T^{-1}\left(\begin{bmatrix} 0 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

3. Consider the following matrix
$$A = \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues of A.

Solution:
$$0 = \begin{vmatrix} 2-\lambda & -2 \\ -3 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 6 = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$$
The eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -1$.

(b) Find the eigenvectors corresponding to the eigenvalues in part (a).

Solution: Using eigenvalue $\lambda_1 = 4$, we derive
$$\begin{bmatrix} -2 & -2 \\ -3 & -3 \end{bmatrix} \xrightarrow{R_1 \to -\frac{1}{2}R_1} \begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 3R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$
A solution to this is $x = 1, y = -1$ yielding the eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Next, using eigenvalue $\lambda_1 = -1$, we derive
$$\begin{bmatrix} 3 & -2 \\ -3 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix}$$
A solution to this is $x = 2, y = 3$ yielding the eigenvector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(c) Give a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

Solution: Using the results of part (b), we can choose
$$D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}.$$

(d) Compute P^{-1} and verify that $A = PDP^{-1}$ (show your steps).

Solution: By the inverse for 2×2 matrices, $P^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}$.
$$PD = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -4 & -3 \end{bmatrix}.$$
$$PDP^{-1} = \begin{bmatrix} 4 & -2 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix} = A.$$