MATH 1500 D01 Fall 2016 Assignment 1

SHOW ALL WORK to get full marks. Leave answers as a fraction. For example, leave it as fractions such as 1/7 as opposed to decimals such as 0.142857. Word problems should have sentence answers with units. Fractions should be lowest terms.

All assignments must be handed in on UMLearn as **one PDF file**. Late assignments will not be accepted. Failure to follow the instructions will result in a mark of 0.

Techniques from this course must be used to solve the questions, not more advanced techniques. For example L'Hopital's Rule for solving limits is not permitted.

The assignment covers sections 1.1, 1.3, 1.5, 2.2, 2.3, 2.5 and 2.6 in the textbook.

1. For the following functions, simplify $\dfrac{f(a+h)-f(a)}{h}$ as much as possible. (At minimum, neither answer should have a **factor** of h when you are done.)

[4] (a) $f(x) = x^2 - 4x + 3$

[4] (b) $f(x) = \dfrac{1}{\sqrt{5x+3}}$

Solution:

(a)

$$\begin{aligned}
\frac{f(a+h)-f(a)}{h} &= \frac{\big((a+h)^2 - 4(a+h) + 3\big) - \big(a^2 - 4a + 3\big)}{h} \\
&= \frac{\big(a^2 + 2ah + h^2 - 4a - 4h + 3\big) - \big(a^2 - 3a + 5\big)}{h} \\
&= \frac{2ah + h^2 - 4h}{h} \\
&= 2a + h - 4
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{f(a+h)-f(a)}{h} &= \frac{\dfrac{1}{\sqrt{5(a+h)+3}} - \dfrac{1}{\sqrt{5a+3}}}{h} \\
&= \frac{\dfrac{\sqrt{5a+3}-\sqrt{5a+5h+3}}{\sqrt{5a+5h+3}\sqrt{5a+3}}}{h} \\
&= \frac{\left(\sqrt{5a+3}-\sqrt{5a+5h+3}\right)\left(\sqrt{5a+3}+\sqrt{5a+5h+3}\right)}{h\sqrt{5a+5h+3}\sqrt{5a+3}\left(\sqrt{5a+3}+\sqrt{5a+5h+3}\right)} \\
&= \frac{5a+3-(5a+5h+3)}{h\sqrt{5a+5h+3}\sqrt{5a+3}\left(\sqrt{5a+3}+\sqrt{5a+5h+3}\right)} \\
&= \frac{-5h}{h\sqrt{5a+5h+3}\sqrt{5a+3}\left(\sqrt{5a+3}+\sqrt{5a+5h+3}\right)} \\
&= \frac{-5}{\sqrt{5a+5h+3}\sqrt{5a+3}\left(\sqrt{5a+3}+\sqrt{5a+5h+3}\right)}
\end{aligned}$$

2. For the function f defined by $f(x) = x^2 - 6x + 5$:

[2] (a) Convert the function to the form $f(x) = (x-h)^2 + k$.

[3] (b) Find an interval (as large as possible) such that f is one–to–one.

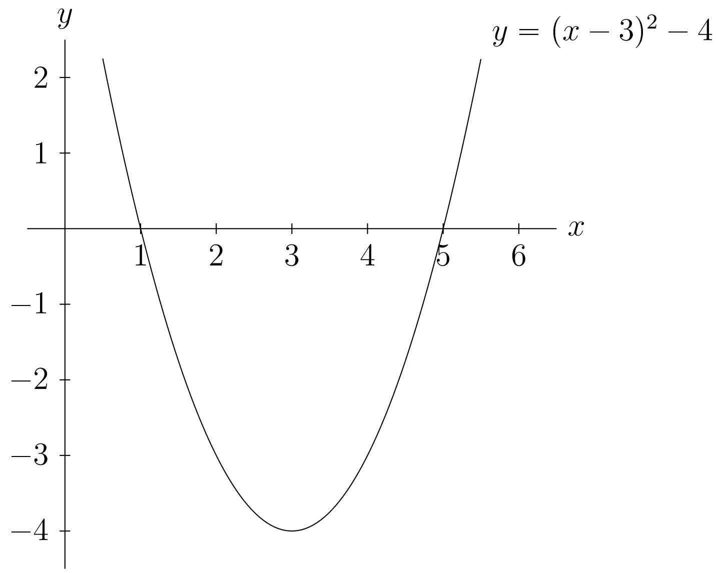
[2] (c) Find the inverse of f on the interval from part (b).

[3] (d) State the domain and range of f and f^{-1} using the restriction from part (b).

Solution:

(a) Completing the square leads to $x^2-6x+5 = (x^2-6x+9)+5-9 = (x-3)^2-4$.

(b) A sketch of the graph yields



Note that this function is not one–to–one as two different x value can lead to the same y value (for example $x = 2$ and $x = 4$ both yield $y = -3$.

However if we just take the part when $x \geqslant 3$, then the function is one–to–one as the function is always increasing and thus can't repeat itself.

Note: We could also take $x \leqslant 3$ as get the same argument.

(c) Solving for x when $x \geqslant 3$ yields

$$y = (x-3)^2 - 4 \Rightarrow (x-3)^2 = y + 4 \Rightarrow x - 3 = \sqrt{y+4} \Rightarrow x = 3 + \sqrt{y+4}.$$

Thus $f^{-1}(x) = 3 + \sqrt{x+4}$.

Note: We only take the positive square root because $x \geqslant 3 \Rightarrow x - 3 \geqslant 0$. If we had taken $x \leqslant 3$ in part b, we would get $f^{-1}(x) = 3 - \sqrt{x+4}$.

(d) The domain of f is $(3, \infty)$ which means the range of f^{-1} is also $(3, \infty)$. The domain of f^{-1} can be found by noting that you cant take the square root of a negative. Hence $y \geqslant -4$. Thus the domain of f^{-1} and the range of f are $(-4, \infty)$

Note: If we had taken $x \leqslant 3$ in part b, then the domain of f and the range of f^{-1} would be $(-\infty, 3)$. The other two would be unchanged.

3. Solve the following equations. If there are any logarithms in the final answer, they should be the natural logarithm.

[4] (a) $\log_3(5 - 2x) + \log_3(x + 5) = 3$

[5] (b) $2^{x+6} = 3\left(5^{4x+1}\right)$

[3] (c) $e^{2x} - 3e^{x} - 10 = 0$

Solution:

(a)

$$\begin{aligned}
\log_3(5-2x) + \log_3(x+5) &= 3 \\
\log_3\left((5-2x)(x+5)\right) &= 2 \\
(5-2x)(x+5) &= 3^3 \\
-2x^2 - 5x + 25 &= 27 \\
2x^2 + 5x + 2 &= 0 \\
(2x+1)(x+2) &= 0 \\
x &= -1/2,\ -2
\end{aligned}$$

Checking both answers yields both answers as solutions since both sides only take the logarithm of positive numbers.

(b) Taking the natural logarithm of both sides yields

$$\begin{aligned}
\ln 2^{x+6} &= \ln 3\left(5^{4x+1}\right) \\
\ln 2^{x+6} &= \ln 3 + \ln 5^{4x+1} \\
(x+6)\ln 2 &= \ln 3 + (4x+1)\ln 5 \\
x\ln 2 + 6\ln 2 &= \ln 3 + 4x\ln 5 + \ln 5 \\
6\ln 2 - \ln 3 - \ln 5 &= 4x\ln 5 - x\ln 2 \\
6\ln 2 - \ln 3 - \ln 5 &= x(4\ln 5 - \ln 2) \\
x &= \frac{6\ln 2 - \ln 3 - \ln 5}{4\ln 5 - \ln 2}
\end{aligned}$$

(c)

$$e^{2x} - 3e^{x} - 10 = 0 \Rightarrow (e^x - 5)(e^x + 2) = 0 \Rightarrow e^x = -2,\ 5.$$

Since the exponential e^x is always positive, this means $e^x = 5 \Rightarrow x = \ln 5$.

[3] 4. (a) Sketch the graph

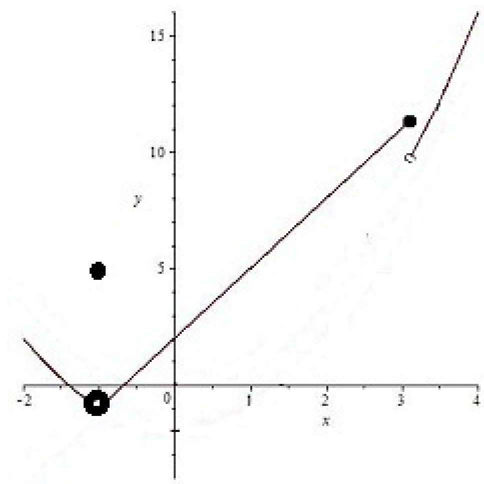
$$f(x) = \begin{cases} x^2 - 2 & x < -1 \\ 3 & x = -1 \\ 3x + 2 & -1 < x \leqslant 3 \\ x^2 & x > 3 \end{cases}$$

[4] (b) Find the following limits if they exists. If they don't exist, explain why

$$\lim_{x\to -1^-} f(x) \quad \lim_{x\to -1^+} f(x) \quad \lim_{x\to -1} f(x) \quad \lim_{x\to 0} f(x) \quad \lim_{x\to 3^-} f(x) \quad \lim_{x\to 3^+} f(x) \quad \lim_{x\to 3} f(x)$$

[3] (c) Is the function continuous at the values $x = -1, x = 0$ and $x = 3$. Explain your answer.

Solution:

(a) 

(b) $\lim_{x\to -1^-} f(x) = -1$

$\lim_{x\to -1^+} f(x) = -1$

$\lim_{x\to -1} f(x) = -1$

$\lim_{x\to 0} f(x) = 2$

$\lim_{x\to 3^-} f(x) = 11$

$\lim_{x\to 3^+} f(x) = 9$

$\lim_{x\to 3} f(x)$ does not exist since the left and right hand limits are not the same.

(c) The function is not continuous at $x = -1$ since the function $f(-1) = 3$ exists is not the same as the limit $\lim_{x\to -1} f(x) = -1$.

The function is continuous at $x = 0$ since the function $f(0) = 2$ exists, the limit $\lim_{x\to 0} f(x) = 2$ exists and they are equal to each other.

The function is not continuous at $x = 3$ since the limit $\lim_{x\to 3} f(x)$ does not exist.

5. Find the following limits if they exist. If they don't exist, determine whether the limit is ∞, $-\infty$ or neither.

[1] (a) $\lim_{x\to 1} \dfrac{x^2+3x}{x^2+4x+5}$

[3] (b) $\lim_{x\to 1^+} \dfrac{x^3-2x^2-4x+5}{x^2-1}$ (Hint: If $x = 1$ makes a polynomial equal to 0 then what must be a factor?)

[4] (c) $\lim_{x\to 3} \dfrac{\sqrt{x-2}-\sqrt{4-x}}{x^2+2x-15}$

[4] (d) $\lim_{x\to 2^+} \dfrac{x^2-5x+6}{x^2-4x+4}$

[4] (e) $\lim_{x\to 3} \dfrac{x^2+2x-15}{|6-2x|}$

[3] (f) $\lim_{x\to 0^+} x\sin\left(\dfrac{2}{x^2}\right)$

[4] (g) $\lim_{x\to -\infty} \dfrac{2x+3}{7x+\sqrt{x^2-2x}}$

[3] (h) $\lim_{x\to \infty} \dfrac{1}{x-\sqrt{x^2-5x}}$

Solution:

(a) $\lim_{x\to 1} \dfrac{x^2+3x}{x^2+4x+5} = \dfrac{(1)^2+3(1)}{(1)^2+4(1)+5} = \dfrac{2}{5}.$

(b) $\lim_{x\to 1^+} \dfrac{x^3-2x^2-4x+5}{x^2-1}$ is in 0/0 form. Thus plugging in $x = 1$ means both polynomials are zero. Hence $x-1$ factors both the numerator and the denominator.

$$\lim_{x\to 1^+} \frac{x^3-2x^2-4x+5}{x^2-1} = \lim_{x\to 1^+} \frac{(x-1)(x^2-x-5)}{(x-1)(x+1)} = \lim_{x\to 1^+} \frac{x^2-x-5}{x+1} = -\frac{5}{2}.$$

(c) $\lim_{x\to 3} \dfrac{\sqrt{x-2}-\sqrt{4-x}}{x^2+2x-15}$ is in 0/0 form. Since we have square roots, we won't be able to find the factor of $x-2$ in the numerator until we multiply by the conjguate.

$$
\begin{aligned}
\lim_{x\to 3}\frac{\sqrt{x-2}-\sqrt{4-x}}{x^2+2x-15} &= \lim_{x\to 3}\frac{\left(\sqrt{x-2}-\sqrt{4-x}\right)\left(\sqrt{x-2}+\sqrt{4-x}\right)}{(x+5)(x-3)\left(\sqrt{x-2}+\sqrt{4-x}\right)}\\
&= \lim_{x\to 3}\frac{x-2-(4-x)}{(x+5)(x-3)\left(\sqrt{x-2}+\sqrt{4-x}\right)}\\
&= \lim_{x\to 3}\frac{2x-6}{(x+5)(x-3)\left(\sqrt{x-2}+\sqrt{4-x}\right)}\\
&= \lim_{x\to 3}\frac{2}{(x+5)\left(\sqrt{x-2}+\sqrt{4-x}\right)}\\
&= \frac{2}{(8)(2)}\\
&= \frac{1}{8}.
\end{aligned}
$$

(d) $\displaystyle\lim_{x\to 2^+}\frac{x^2-5x+6}{x^2-4x+4}$ is in 0/0 form, so factoring leaves it in

$$
\lim_{x\to 2^+}\frac{(x-3)(x-2)}{(x-2)^2} = \lim_{x\to 2^+}\frac{x-3}{x-2}
$$

which is in non–zero/0 form. Hence the limit goes to either $\pm\infty$.

The numerator is negative when $x = 2$ and as $x \to 2^+$ the denominator is positive. Hence the limit is $-\infty$.

(e) $\displaystyle\lim_{x\to 3}\frac{x^2+2x-21}{|6-2x|}$ is in 0/0 form. However the $|6-2x|$ is a piecewise function which changes at $x = 3$. Thus we must take one-sided limits.

$$
\lim_{x\to 3^+}\frac{x^2+2x-15}{|6-2x|} = \lim_{x\to 3^+}\frac{(x-3)(x+5)}{-(6-2x)} = \lim_{x\to 3^+}\frac{x+5}{2} = 4
$$

$$
\lim_{x\to 3^-}\frac{x^2+2x-15}{|6-2x|} = \lim_{x\to 3^-}\frac{(x-3)(x+5)}{6-2x} = \lim_{x\to 3^-}\frac{x+5}{-2} = -4
$$

Since the left and right hand limits are not the same, the limit does not exist, and is neither ∞ nor $-\infty$.

(f) $\displaystyle\lim_{x\to 0^+} x\sin\left(\frac{2}{x^2}\right)$ doesn't follow the limits laws as the limit of $\sin\left(\dfrac{2}{x^2}\right)$ does not exist. However, we can use the squeeze theorem.

Using that $\sin\left(\dfrac{2}{x^2}\right)$ is bounded between -1 and 1, we know that

$$-x \leqslant x \sin\left(\frac{2}{x^2}\right) \leqslant x$$

for $x > 0$. Since

$$\lim_{x\to 0^+} -x = \lim_{x\to 0^+} x = 0,$$

by the squeeze theorem we know that $\displaystyle\lim_{x\to 0^+} x \sin\left(\frac{2}{x^2}\right) = 0$.

(g) Factoring out the largest power of x which is either x or $\sqrt{x^2} = |x|$ yields

$$\begin{aligned}
\lim_{x\to -\infty} \frac{2x+3}{7x+\sqrt{x^2-2x}} &= \lim_{x\to -\infty} \frac{x(2+3/x)}{7x+|x|\sqrt{1-2/x}} \\
&= \lim_{x\to -\infty} \frac{x(2+3/x)}{7x-x\sqrt{1-2/x}} \\
&= \lim_{x\to -\infty} \frac{2+3/x}{7-\sqrt{1-2/x}} \\
&= \frac{1}{3}
\end{aligned}$$

(h) The denominator is of $\infty - \infty$ form which is indeterminant. Therefore we want to multiply by the conjugate

$$\begin{aligned}
\lim_{x\to\infty} \frac{1}{x-\sqrt{x^2-5x}} &= \lim_{x\to\infty} \frac{x+\sqrt{x^2-5x}}{(x-\sqrt{x^2-5x})(x+\sqrt{x^2-5x})} \\
&= \lim_{x\to\infty} \frac{x+\sqrt{x^2-5x}}{5x} \\
&= \lim_{x\to\infty} \frac{x(1+\sqrt{1-5/x})}{5x} \\
&= \lim_{x\to\infty} \frac{1+\sqrt{1-5/x}}{5} \\
&= \frac{1+\sqrt{1}}{5} \\
&= \frac{2}{5}
\end{aligned}$$

[5] 6. Use limits to calculate a and b such that the following function is continuous for all real numbers x.

$$f(x) = \begin{cases} x^2 + ax + b & x < 3 \\ 16 & x = 3 \\ \ln(x-2) + (a-b)x - 11 & x > 3 \end{cases}$$

Solution:

The function is continuous for $x < 3$ since x^2+ax+b is a polynomial and polynomials are always continuous. The function is continuous for $x < 3$ since $\ln(x-2)+(a-b)x+7$ is the addition of a logarithm and a polynomial and both are always continuous as long as we take the logarithm of a positive value. Hence we focus at $x = 3$

For the function to be continuous at $x = 3$ we must have that

$$\lim_{x \to 3^-} f(x) = f(3) = \lim_{x \to 3^+} f(x)$$

From $\lim_{x \to 3^-} f(x) = f(3)$ we have that

$$\begin{aligned} \lim_{x \to 3^-} f(x) &= f(3) \\ \lim_{x \to 3^-} (x^2 + ax + b) &= 16 \\ 9 + 3a + b &= 16 \\ 3a + b &= 7. \end{aligned}$$

From $\lim_{x \to 3^+} f(x) = f(3)$ we have that

$$\begin{aligned} \lim_{x \to 3^+} f(x) &= f(3) \\ \lim_{x \to 3^-} (\ln(x-2) + (a-b)x - 11) &= 16 \\ \ln 1 + 3(a-b) - 11 &= 16 \\ 3a - 3b &= 27. \\ b &= a - 9. \end{aligned}$$

Since $b = a - 9$. Inserting that into the first equation yields

$$3a + (a - 9) = 7 \Rightarrow 4a = 16 \Rightarrow a = 4 \Rightarrow b = -5.$$

Hence $a = 4$ and $b = -5$.

[4] 7. Show that $x^5 = 3 - 4x$ has a solution on the interval $(0, 1)$. Justify your answer.

Solution: Let $f(x) = x^5 + 4x - 3$. We are trying to show $f(x) = 0$ has a solution on the interval $(0, 1)$. Since f is the addition of a exponential and a polynomial, f is continuous and therefore the Intermediate Value Theorem applies.

$f(0) = -3 < 0$.

$f(1) = 2 > 0$.

Therefore by the Intermediate Value Theorem there exists a value c between 0 and 1 such that $f(c) = 0$.

This assignment is out of 75 points.