MATH 1500 D01 Fall 2016 Assignment 2

SHOW ALL WORK to get full marks. Leave answers as exact answers. For example, leave it as 1/7 as opposed to 0.142857. Word problems should have sentence answers with units. This assignment covers sections from 2.7, 2.8, 3.1–3.5, 3.9.

Use the definition of derivative only for Q1. Do not use the definition in any other questions.

Word problems should have sentence answers with units.

[7] 1. Use the definition of the derivative to calculate the derivative of the function $f(x) = \dfrac{1}{\sqrt{x^2+5}}$.

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{(x+h)^2+5}} - \dfrac{1}{\sqrt{x^2+5}}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x^2+5}-\sqrt{(x+h)^2+5}}{h\sqrt{x^2+5}\sqrt{(x+h)^2+5}} \\
&= \lim_{h\to 0} \frac{\left(\sqrt{x^2+5}-\sqrt{(x+h)^2+5}\right)\left(\sqrt{x^2+5}+\sqrt{(x+h)^2+5}\right)}{h\sqrt{x^2+5}\sqrt{(x+h)^2+5}\left(\sqrt{x^2+5}+\sqrt{(x+h)^2+5}\right)} \\
&= \lim_{h\to 0} \frac{(x^2+5)-((x+h)^2-5)}{h\sqrt{x^2+5}\sqrt{(x+h)^2+5}\left(\sqrt{x^2+5}+\sqrt{(x+h)^2+5}\right)} \\
&= \lim_{h\to 0} \frac{-2xh-h^2}{h\sqrt{x^2+5}\sqrt{(x+h)^2+5}\left(\sqrt{x^2+5}+\sqrt{(x+h)^2+5}\right)} \\
&= \lim_{h\to 0} \frac{-2x-h}{\sqrt{x^2+5}\sqrt{(x+h)^2+5}\left(\sqrt{x^2+5}+\sqrt{(x+h)^2+5}\right)} \\
&= \frac{-2x-0}{\sqrt{x^2+5}\sqrt{3(x+(0))+5}\left(\sqrt{x^2+5}+\sqrt{(x+0)^2+5}\right)} \\
&= \frac{-x}{(\sqrt{x^2+5})^3}
\end{aligned}$$

2. Calculate the derivative of the following functions. Do not simplify your answers.

[3] (a) $f(x) = \sqrt[3]{x} + \dfrac{2}{x^{2016}} + \dfrac{1}{\sqrt{x^3}}$

[4] (b) $f(x) = \dfrac{4 - xe^x}{x + e^x}$

[5] (c) $f(x) = e^{e^{5\tan x}}$

[5] (d) $f(x) = \cos(\sqrt{e^{2x} + \cot x})$

Solution:

(a) $f(x) = x^{1/3} + 2x^{-2016} + x^{-3/2}$

Therefore

$$f'(x) = \frac{1}{3}x^{-2/3} - 4032x^{-2017} - \frac{3}{2}x^{-5/2}$$

(b)

$$\begin{aligned} f'(x) &= \frac{(4 - xe^x)'\,(x + e^x) - (4 - xe^x)(x + e^x)'}{(x + e^x)^2} \\ &= \frac{(-(xe^x + e^x))(x + e^x) - (4 - xe^x)(1 + e^x)}{(x + e^x)^2} \end{aligned}$$

(c)

$$\begin{aligned} f'(x) &= e^{e^{5\tan x}} \cdot \left(e^{5\tan x}\right)' \\ &= e^{e^{5\tan x}} \cdot e^{5\tan x}(5\tan x)' \\ &= e^{e^{5\tan x}} \cdot e^{5\tan x} \cdot 5\sec^2 x \end{aligned}$$

(d)

$$\begin{aligned} f'(x) &= -\sin(\sqrt{e^{2x} + \cot x})(\sqrt{e^{2x} + \cot x})' \\ &= -\sin(\sqrt{e^{2x} + \cot x})\frac{1}{2}(e^{2x} + \cot x)^{-1/2}(e^{2x} + \cot x)' \\ &= -\frac{1}{2}\sin(\sqrt{e^{2x} + \cot x})(e^{2x} + \cot x)^{-1/2}(2e^{2x} - \csc^2 x) \end{aligned}$$

3. Evaluate the following limits. Show all steps.

[4] (a) $\lim_{x \to 0} \frac{2 \sin 3x}{\sin 5x}$

[4] (b) $\lim_{t \to 2} \frac{3 \sin(t-2)}{(t^2 - 6t + 8)}$

Solution: We will repeatedly use that $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$.

(a)

$$\begin{aligned} \lim_{x \to 0} \frac{2 \sin 3x}{\sin 5x} &= 2 \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot \frac{5x}{\sin 5x} \cdot \frac{3x}{5x} \\ &= \frac{6}{5} \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot \frac{1}{\lim_{x \to 0} \frac{\sin 5x}{5x}} \\ &= \frac{6}{5} \cdot 1 \cdot \frac{1}{1} \\ &= \frac{6}{5} \end{aligned}$$

(b) Suppose $\theta = t - 2$ Then $\theta \to 0$ as $t \to 2$. Hence the limit is

$$\begin{aligned} \lim_{t \to 2} \frac{3 \sin(t-2)}{(t^2 - 6t + 8)} &= \lim_{t \to 2} \frac{2 \sin(t-2)}{(t-2)(t-4)} \\ &= 3 \lim_{t \to 2} \frac{\sin(t-2)}{t-2} \cdot \lim_{t \to 2} \frac{1}{t-4} \\ &= 3 \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \cdot \lim_{t \to 2} \frac{1}{t-4} \\ &= 3 \cdot 1 \cdot \frac{1}{2-4} \\ &= -\frac{3}{2} \end{aligned}$$

Note: The substitution could be used to change the entire limit to θ and obtain the new limit

$$\lim_{\theta \to 0} \frac{3 \sin \theta}{\theta^2 - 2\theta}.$$

Afterwards, the steps would follow similarly.

4. If a rock is thrown upwards with a velocity of 15 metres per second, its height, in metres, is given by

$$H(t) = -4.9t^2 + 15t.$$

(Calculus method and not physics methods must be used)

[1] (a) Find the velocity at any time t seconds.

[3] (b) After how many seconds will the rock stop rising and begin to fall?

[4] (c) After what time, and with what velocity will the rock hit the ground?

Solution:

(a) The velocity is just the derivative.

$v(t) = H'(t) = -9.8t + 15.$

(b) This happens when the velocity is 0. Hence

$$0 = -9.8t + 15 \Rightarrow t = \frac{15}{9.8}$$

Therefore the rock will stop rising and being to fall at $\frac{15}{9.8}$ seconds.

(c) The ball will hit the ground when the height is 0. Therefore

$$0 = H(t) = -4.9t^2 + 15t = t(-4.9t + 15) \Rightarrow t = 0,\ t = \frac{15}{4.9}$$

Since $t = 0$ is when the rock is thrown into the ball, it will hit the ground in $\frac{15}{4.9}$ seconds.

Plugging in $a = \frac{15}{4.9}$ into the velocity function yields

$$v\left(\frac{15}{4.9}\right) = -9.8\left(\frac{15}{4.9}\right) + 15 = -30 + 15 = -15.$$

Therefore the velocity when the ball hits the ground will be -15 metres per second (or 15 metres per second in the downward direction.)

[5] 5. Calculate an equation of the tangent line to $f(x) = (x^3 - 2x - 1)e^x$ when $x = 0$.

Solution:

Finding the y- coordinate yields $x = 0 \Rightarrow y = f(0) = (0 + 0 - 1)e^0 = -1$.

$f'(x) = (x^3 - 2x - 1)' e^x = (x^3 - 2x - 1)(e^x)' = (3x^2 - 2)e^x + (x^3 - 2x - 1)e^x$ Hence $f'(0) = (0 - 2)e^0 + (0 + 0 - 1)e^0 = -3$.

Hence an equation is

$$y + 1 = -3(x - 0) \text{ or } y = -3x - 1.$$

[4] 6. If $f(2) = 4$ and $f'(2) = -3$, calculate $g'(2)$ if $g(x) = \dfrac{f(x)}{x}$.

Solution:

$g'(x) = \dfrac{xf'(x) - f(x)}{x^2}$.

Therefore $g'(2) = \dfrac{2f'(2) - f(2)}{2^2} = \dfrac{2(-3) - 4}{2^2} = -\dfrac{5}{2}$.

[8] 7. Use implicit differentiation to find the equation of the tangent line to the curve

$$x^3 - \tan xy + y^3 = e^y - 1$$

at the point $(1, 0)$.

Solution:

Method 1: Find dy/dx in general

Taking the implicit derivative obtains

$$3x^2 - \sec^2(xy)(y + xy') + 3y^2y' = e^y y'$$

Simplifying:

$$3x^2 - y\sec^2(xy) + x\sec^2(xy)y' + 3y^2y' = e^y y'$$

Isolating y'

$$3x^2 - y\sec^2(xy) = x\sec^2(xy)y' - 3y^2y' + e^y y'$$

Factor y' and divide

$$y' = \frac{3x^2 - y\sec^2(xy)}{x\sec^2(xy) - 3y^2 + e^y}$$

The slope at the point (1, 0) is therefore

$$m = y'|_{(x,y)=(1,0)} = \frac{3(1)^2 - 0\sec^2(0)}{\sec^2(0) - 0 + e^0} = \frac{3}{2}$$

Hence the equation of the tangent line is $y = \frac{3}{2}(x-1)$.

Method 2: Just find y' at the given point

Taking the implicit derivative obtains

$$3x^2 - \sec^2(xy)(y + xy') + 3y^2y' = e^y y'$$

Plugging in $(x, y) = (1, 0)$ and letting $m = y'|_{(x,y)=(1,0)}$ yields

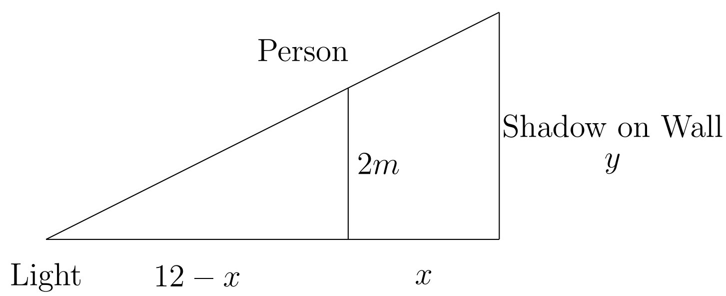
$$3 - \sec^2(0)(0 + m) + 0 = e^0 m \Rightarrow 3 - m = m \Rightarrow 3 = 2m \Rightarrow m = \frac{3}{2}.$$

Hence the equation of the tangent line is $y = \frac{3}{2}(x-1)$.

[9] 8. A spotlight on the ground shines on a wall 12 metres away. A person 2 metres tall walks from the spotlight towards the building at a speed of 4/5 metres per second. How fast is the length of their shadow on the building changing when they are 4 meters from the building?

Solution:

A picture is below:



Let y be the length of the shadow at any time and x be the distance between the person and the wall at any time. (Note: It can also be done similarly if x is the distance from the light and $12 - x$ is the distance to the wall)

$dx/dt = -4/5$ m/s and we want to compute dy/dt when $x = 4$ m.

To obtain an equation, we can use the two similar triangles.

$$\frac{y}{12} = \frac{2}{12 - x} \Rightarrow y = \frac{24}{12 - x}$$

Taking the derivative with respect to time yields:

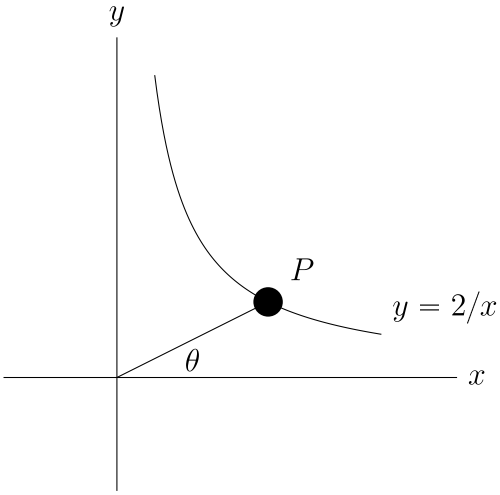
$$\frac{dy}{dt} = \frac{24}{(12 - x)^2} \frac{dx}{dt}$$

At the given point and rate for x this yields

$$\frac{dy}{dt} = \frac{24}{8^2} \cdot \frac{-4}{5} = -\frac{3}{10}.$$

Hence the length of the shadow is decreasing at a rate of 3/10 m/s.

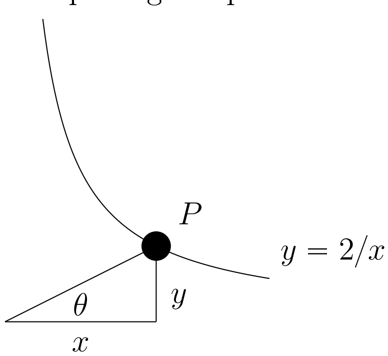
[9] 9. A particle P is moving along the curve $y = \dfrac{2}{x}$ so that its x coordinate (in meters) is increasing at a rate of 2 m/s. The line segment between the origin (0, 0) and P forms an angle θ between the line segment and the positive x-axis. See the figure below



Compute the rate of change of θ with respect to time when the particle passes through $x = 2$. (Hint: You may wish to use $\sec^2\theta = 1 + \tan^2\theta$ at some point.)

Solution:

Completing the picture



We know $\dfrac{dx}{dt} = 2m$ and we want to find $\dfrac{d\theta}{dt}$ when $x = 2m$

Thus we have the equation $\tan\theta = \dfrac{y}{x} = \dfrac{2}{x^2}$.

Taking the derivative yields

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{-4}{x^3}\frac{dx}{dt}$$

At $x = 2$ we have that $\tan\theta = \dfrac{1}{2}$ and thus $\sec^2\theta = 1 + \dfrac{1}{4} = \dfrac{5}{4}$.

Hence at the given point we have

$$\frac{5}{4} \cdot \frac{d\theta}{dt} = \frac{-4}{8}(2) \Rightarrow \frac{d\theta}{dt} = -\frac{4}{5}.$$

Hence at $x = 2m$ we have that the angle is decreasing at a rate of 4/5 radians per second.

This assignment is out of 75 points.