Math 1500 D01 Fall 2017
Written Assignment #1 Solutions

1. For each of the following functions $f(x)$, simplify the difference quotient $\dfrac{f(a+h)-f(a)}{h}$ as far as possible. In (a), (b) it is expected that h will not appear as a factor in either the numerator or denominator of the resulting expression.

[3] (a) $f(x) = x^3 + x$

[3] (b) $2\sqrt{x+3}$

[3] (c) $2^x + 1$

Solution

(a) $\dfrac{(a+h)^3+(a+h)-(a^3+a)}{h} = \dfrac{3a^2h+3ah^2+h^3+h}{h} = 3a^2+3ah+h^2$

(b) $\dfrac{2\sqrt{a+h+3}-2\sqrt{a+3}}{h} = \dfrac{2(\sqrt{a+h+3}-\sqrt{a+3})(\sqrt{a+h+3}+\sqrt{a+3})}{h(\sqrt{a+h+3}+\sqrt{a+3})} = \dfrac{2(a+h+3-(a+3))}{h(\sqrt{a+h+3}+\sqrt{a+3})}$

$= \dfrac{2h}{h(\sqrt{a+h+3}+\sqrt{a+3})} = \dfrac{2}{\sqrt{a+h+3}+\sqrt{a+3}}$

(c) $\dfrac{2^{a+h}+1-(2^a+1)}{h} = 2^a\dfrac{(2^h-1)}{h}$

[5] 2. Find the domain of $p(x) = \dfrac{x^2 - 3x}{x^2 - 5x + 6} + \sqrt{6 - x}$. Report your answer in interval form.

Solution

For x to be in the domain of $p(x)$ we must be able to evaluate the radical, so $x \leq 6$. Further, we cannot divide by $x^2 - 5x = (x-2)(x+3)$ and so we cannot have $x = 2$ or -3.

Therefore the domain of $p(x)$ is $(-\infty, -3) \bigcup (-3, 2) \bigcup (2, 6]$.

3. Define a function by $f(x) = x^2 - 8x + 5$.

[2] (a) Rewrite this function in the form $f(x) = (x-h)^2 + k$.

[2] (b) Sketch a graph of this function by comparing $y = f(x)$ to $y = x^2$.

[2] (c) Find an interval, as large as possible, on which f is a one-to-one function.

[2] (d) Write down the inverse of f on the interval you found in (c) above.

Solution

(a) $f(x) = (x-4)^2 - 4^2 + 5 = (x-4)^2 + (-11)$. (i.e., $h = 4$, $k = -11$)

(b) I won't sketch it here but I'll describe it. Start with the standard parabola $y = x^2$. Replacing x with $x - 4$ moves the graph 4 units to the right. Adding -11 to the function moves the graph downward by 11 units. So you should have a congruent parabola to the original, but with vertex moved to $(4, -11)$. It should cross the x axis around 1.5 and 6.5 (as the quadratic formula will show).

(c) f is increasing on $[4, \infty)$, and on this interval $x - 4 \geq 0$.

(d) A standard way to find the inverse is to set $y = f(x)$ and reverse the roles of x and y (which is merely another way to look at the inverse relation), then solve for y in the new relation. Alternatively, solve for x in $y = f(x)$ then interpret the resulting relation as $x = f^{-1}(y)$ (which is also another way of understanding the definition of an inverse function). Thus $y = (x-4)^2 - 11$ so $(x-4)^2 = y + 11$ so $x - 4 = \sqrt{y+11}$ (here we are using the fact that $x - 4 \geq 0$ on the interval in question). Thus $x = \sqrt{y+11} + 4 = f^{-1}(y)$.
So $f^{-1}(x) = \sqrt{x+11} + 4$ (on domain $x \geq -11$).

4. Solve each of the following equations for x. In your answers, do not use any logarithms except for natural logarithm.

[4] (a) $\log_3(5x+2) - \log_9(2x+1) = -1$

[4] (b) $2^{x+1} = \sqrt{3^{2x-1}}$

[4] (c) $e^{2x+2} + 6 = 5e^{x+1}$

Solution

(a) $\log_9(2x+1) = \frac{1}{2}\log_3(2x+1)$, so we have

$$3^{\log_3(5x+2)-\frac{1}{2}\log_3(2x+1)} = 3^{-1}$$

so

$$\frac{5x+2}{\sqrt{2x+1}} = \frac{1}{3}.$$

Clearing the fractions, squaring and gathering terms gives $225x^2+178x+35=0$. The quadratic formula yields $x = \frac{-89\pm\sqrt{46}}{225} < 0$. Since we must have $2x+5>0$ for the first term of the given expression to be evaluated, there are <u>no solutions</u>.

(b) We have $\ln\left(2^{x+1}\right) = \ln\left(\left(3^{2x-1}\right)^{\frac{1}{2}}\right) = \ln 3^{x-\frac{1}{2}}$. So $(x+1)\ln 2 = (x-\frac{1}{2})\ln 3$. Thus

$$x(\ln 2 - \ln 3) = -\ln 2 - \frac{1}{2}\ln 3$$

$$x = \frac{-\ln 2 - \frac{1}{2}\ln 3}{\ln 2 - \ln 3} = \frac{\ln 3 + 2\ln 2}{2(\ln 3 - \ln 2)} = \frac{\ln 12}{2\ln\frac{3}{2}}$$

(for one—there are a number of more-or-less equivalently simplified ways to write this expression)

(c) $\left(e^{x+1}\right)^2 - 5e^{x+1} + 6 = 0 = \left(e^{x+1}-2\right)\left(e^{x+1}-3\right)$. Therefore $e^{x+1} = 2$ or 3. Taking logs we obtain two solutions, $x = (\ln 2) - 1$ or $(\ln 3) - 1$.

5. Let $f(x) = \begin{cases} x+1 & x \leq -1 \\ x^2 & -1 < x < 0 \\ 3x & 0 < x < 1 \\ 5 & x = 1 \\ 2-x & x > 1 \end{cases}$

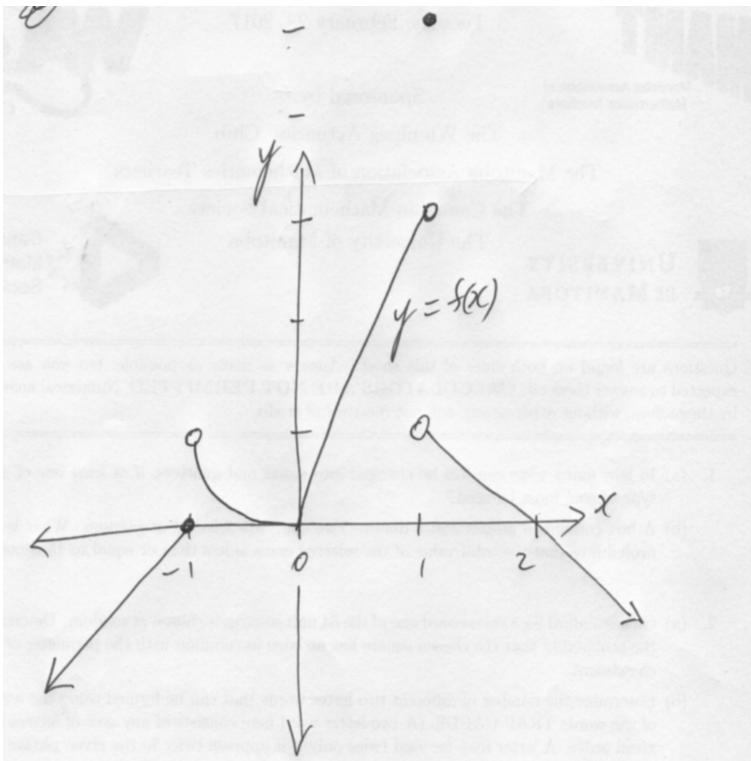
(a) Roughly sketch a graph of $y = f(x)$, being sure to indicate whether points are included or excluded from the
[3] curve at ends of subdomain intervals.

[6] (b) Find $\lim_{x \to a^+} f(x)$, $\lim_{x \to a^-} f(x)$, $\lim_{x \to a} f(x)$ for each of $a = -1, 1, 2$. If any don't exist, explain.

[3] (c) At which of points $-1, 0, 1$ is $f(x)$ discontinous? Explain your reasoning for each.

Solution

(a)



Note the isolated point at $(1, 5)$

(b) $\lim_{x \to -1^-} f(x) = 0$, $\lim_{x \to a^+} f(x) = 1$, $\lim_{x \to -1} f(x)$ does not exist because the one-sided limits are unequal;
$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0} f(x) = 0$;
$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = \lim_{x \to 2} f(x) = 0$

(c) f is discontinuous at $a = -1$ and 1 because in these cases $\lim_{x \to a}$ does not exist because the two one-sided limits are unequal. f is continuous at 0 because the limit exists and is equal to 0 which is also equal to $f(0)$.

6. Calculate any limits that exist, including any which are infinite limits, and reporting "DNE" otherwise.

[2] (a) $\lim_{x \to \frac{3\pi}{4}} \frac{\sin x}{x}$

[2] (b) $\lim_{x \to \pi^-} \frac{x}{\sin x}$

[4] (c) $\lim_{x \to 2} \frac{x^3 + 2x^2 - 5x - 6}{x^2 - 4}$ (HINT: factor)

[4] (d) $\lim_{x \to 0} \frac{x^2 + 2x}{\sqrt{x^2+9} - \sqrt{x+9}}$ (HINT: rationalize)

[4] (e) $\lim_{t \to -5} \frac{|2t+10|}{t^2 - t - 30}$

[4] (f) $\lim_{x \to 3^+} \frac{2x^2 - 18}{5(x-3)^2}$

[4] (g) $\lim_{x \to 0^+} \frac{1}{x - \sqrt{x^2 + x}}$

Solution

(a) $\frac{\sin \frac{3\pi}{4}}{\frac{3\pi}{4}} = \frac{2\sqrt{2}}{3\pi}$.

(b) DNE (since the denominator $\to 0$ and the numerator $\to \pi$)

(c) $= \lim_{x \to 2} \frac{(x+1)(x-2)(x+3)}{(x-2)(x+2)} = \lim_{x \to 2} \frac{(x+1)(x+3)}{x+2} = \frac{(2+1)(2+3)}{2+2} = \frac{15}{4}$.

(d) $= \lim_{x \to 0} \frac{(x^2+2x)(\sqrt{x^2+9} + \sqrt{x+9})}{(\sqrt{x^2+9} - \sqrt{x+9})(\sqrt{x^2+9} + \sqrt{x+9})} = \lim_{x \to 0} \frac{(x^2+2x)(\sqrt{x^2+9} + \sqrt{x+9})}{x^2 + 9 - (x+9)}$

$= \lim_{x \to 0} \frac{x(x+2)(\sqrt{x^2+9} + \sqrt{x+9})}{x(x-1)} = \lim_{x \to 0} \frac{(x+2)(\sqrt{x^2+9} + \sqrt{x+9})}{x-1} = \frac{(0+2)(\sqrt{0^2+9} + \sqrt{0+9})}{0-1}$

$= -12$

(e) DNE, since the two one-sided limits are not equal: $\lim_{t \to -5^+} \frac{|2t+10|}{t^2 - t - 30} = \lim_{t \to -5^+} \frac{2t+10}{(t-6)(t+5)} = \lim_{t \to -5} \frac{2}{t-6} = -\frac{2}{11}$, while $\lim_{t \to -5^-} \frac{|2t+10|}{t^2 - t - 30} = \lim_{t \to -5^-} \frac{-2t-10}{(t-6)(t+5)} = \lim_{t \to -5} \frac{-2}{t-6} = \frac{2}{11}$

(f) $= \lim_{x \to 3^+} \frac{2(x+3)(x-3)}{5(x-3)^2} = \lim_{x \to 3^+} \frac{2(x+3)}{5(x-3)} = \infty$
(will also accept DNE) since numerator $\to 12$ while denominator $\to 0^+$

(g) $\lim_{x \to 0^+} \frac{x + \sqrt{x^2+x}}{(x - \sqrt{x^2+x})(x + \sqrt{x^2+x})} = \lim_{x \to 0^+} \frac{x + \sqrt{x^2+x}}{x^2 - (x^2+x)} = \lim_{x \to 0^+} \frac{x\left(1 + \sqrt{1 + \frac{1}{x}}\right)}{-x} = \lim_{x \to 0^+} -1 + \sqrt{1 + \frac{1}{x}} = \infty$ (since $\frac{1}{x} \to \infty$ as $x \to 0$)
(NOTE: In this case there are a number of ways to arrive at "DNE" that are either strictly incorrect (in which case a mark of 0 might be attained) or fail to justify the answer (such as arriving at a limit that gives $\frac{0}{0}$ upon "plugging in". That misses the point altogether, and depending on the work, maybe worth at most 2 marks here)

[5] 7. For which values of a and b is the following function continuous everywhere? Show all the steps of your reasoning for full marks; an unjustified correct answer is worth only 1 mark.

$$g(x) = \begin{cases} 2ax + b & x < 1 \\ a^2 + b^2 & x = 1 \\ 7x^3 - (a+b)x + 1 & x > 1 \end{cases}$$

Solution

$\lim\limits_{x \to 1^-} g(x) = 2a + b$; $\lim\limits_{x \to 1^+} g(x) = 8 - (a+b)$; $g(1) = a^2 + b^2$.
Setting these three expressions equal to each other: $2a + b = 8 - a - b = a^2 + b^2$. From the first two, $3a + 2b = 8$, or $b = 4 - \frac{3}{2}a$. Setting the first equal to the third and eliminating b,

$$2a + (4 - \frac{3}{2}a) = a^2 + (4 - \frac{3}{2}a)^2$$

Simplifying we obtain $13a^2 - 50a + 48 = (13a - 24)(a - 2)$ (or, alternatively, just use the quadratic formula to obtain roots) so that the solutions are $a = 2, b = 1$ and $a = \frac{24}{13}, b = \frac{16}{13}$)

[4] 8. Use a result from this course to prove that $g(x) = x^3 - 11x - 21$ has a zero somewhere in the interval $(4, 5)$.

Solution

$f(4) = 64 - 44 - 21 = -1 < 0$ and $f(5) = 125 - 55 - 21 = 49 > 0$. It follows, by the IVT, that $f(c) = 0$ for some $c \in (4, 5)$.