MATH 1500 D01 Winter 2017 Assignment 1

SHOW ALL WORK to get full marks. Leave answers as exact answers. For example, leave it as fractions such as $1/7$ as opposed to decimals such as 0.142857 or $\ln 4$ or e^{15}. Word problems should have sentence answers with units. Fractions should be lowest terms.

Calculators are not permitted. Assignments using a calculator will not be graded.

All assignments must be handed in on UMLearn as **one PDF file**. Late assignments will not be accepted. Failure to follow the instructions will result in a mark of 0.

Techniques from this course must be used to solve the questions, not more advanced techniques. For example L'Hopital's Rule for solving limits is not permitted.

This assignment covers sections 1.1, 1.3, 1.4, 1.5, 2.2, 2.3, 2.5 and 2.6 in the textbook.

1. For the following functions, simplify $\dfrac{f(a+h)-f(a)}{h}$ as much as possible. (At minimum, neither answer should have a **factor** of h when you are done. Further note that this question is not asking for a derivative nor a limit.)

[4] (a) $f(x) = x^2 - 6x + 7$

[4] (b) $f(x) = \dfrac{1}{\sqrt{3x-2}}$

Solution:

(a)

$$\begin{aligned}
\frac{f(a+h)-f(a)}{h} &= \frac{\big((a+h)^2 - 6(a+h) + 7\big) - \big(a^2 - 6a + 7\big)}{h} \\
&= \frac{\big(a^2 + 2ah + h^2 - 6a - 6h + 7\big) - \big(a^2 - 6a + 7\big)}{h} \\
&= \frac{2ah + h^2 - 6h}{h} \\
&= 2a + h - 6
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{f(a+h)-f(a)}{h} &= \frac{\dfrac{1}{\sqrt{3(a+h)-2}} - \dfrac{1}{\sqrt{3a-2}}}{h} \\
&= \frac{\dfrac{\sqrt{3a-2}-\sqrt{3a+3h-2}}{\sqrt{3a+3h-2}\sqrt{3a-2}}}{h} \\
&= \frac{\left(\sqrt{3a-2}-\sqrt{3a+3h-2}\right)\left(\sqrt{3a-2}+\sqrt{3a+3h-2}\right)}{h\sqrt{3a+3h-2}\sqrt{3a-2}\left(\sqrt{3a-2}+\sqrt{3a+3h-2}\right)} \\
&= \frac{3a-2-(3a+3h-2)}{h\sqrt{3a+3h-2}\sqrt{3a-2}\left(\sqrt{3a-2}+\sqrt{3a+3h-2}\right)} \\
&= \frac{-3h}{h\sqrt{3a+3h-2}\sqrt{3a-2}\left(\sqrt{3a-2}+\sqrt{3a+3h-2}\right)} \\
&= \frac{-3}{\sqrt{3a+3h-2}\sqrt{3a-2}\left(\sqrt{3a-2}+\sqrt{3a+3h-2}\right)}
\end{aligned}$$

[4] 2. Compute the domain of $\dfrac{\sqrt{x^2-2x-3}}{2x+3}$. Leave answer in interval notation.

Solution:

There are two restrictions in this question. $x^2-2x-3 \geqslant 0$ and $2x+3 \neq 0 \Rightarrow x \neq -\dfrac{3}{2}$.

As for $x^2-2x-3 \geqslant 0 \Leftrightarrow (x-3)(x+1) \geqslant 0$. For the product to be positive, either both terms must be positive (or 0) and therefore $x \geqslant 3$ or both terms must be negative or 0 and therefore $x \leqslant -1$. Hence the domain is

$$\left(-\infty,\, -\frac{3}{2}\right) \cup \left(-\frac{3}{2},\, -1\right] \cup [3,\, \infty)\,.$$

3. For the function f defined by $f(x) = x^2 + 4x + 5$:

[2] (a) Convert the function to the form $f(x) = (x - h)^2 + k$.

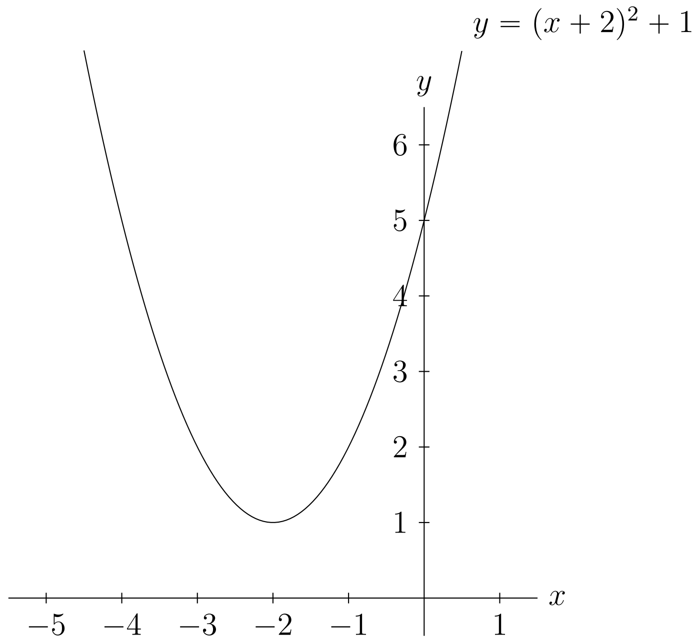
[2] (b) Find an interval such that f is one–to–one.

[2] (c) Find the inverse of f on the interval from part (b).

Solution:

(a) Completing the square leads to $x^2+4x+5 = (x^2+4x+4)-4+5 = (x+2)^2+1$.

(b) A sketch of the graph yields



Note that this function is not one–to–one as two different x value can lead to the same y value (for example $x = -3$ and $x = -1$ both yield $y = 2$.

However if we just take the part when $x \geqslant -2$, then the function is one–to–one as the function is always increasing and thus can't repeat itself.

Note: We could also take $x \leqslant -2$ and use the same argument.

(c) Solving for x when $x \geqslant -2$ yields

$$y = (x+2)^2 + 1 \Rightarrow (x+2)^2 = y - 1 \Rightarrow x + 2 = \sqrt{y-1} \Rightarrow x = -2 + \sqrt{y-1}.$$

Thus $f^{-1}(x) = -2 + \sqrt{x-1}$.

Note: We only take the positive square root because $x \geqslant -2 \Rightarrow x + 2 \geqslant 0$. If we had taken $x \leqslant -2$ in part b, we would get $f^{-1}(x) = -2 - \sqrt{x-1}$.

4. Solve the following equations. If there are any logarithms in the final answer, they should be the natural logarithm.

[4] (a) $\log_5(3-2x)+\log_5(24-x)=3$

[5] (b) $3^{x-1}=4\left(5^{2x+3}\right)$

[3] (c) $e^{2x}-5e^{x}-14=0$

Solution:

(a)

$$\begin{aligned}
\log_5(3-2x)+\log_5(24-x) &= 3\\
\log_5\big((3-2x)(24-x)\big) &= 3\\
(3-2x)(24-x) &= 5^3\\
2x^2-51x+72 &= 125\\
2x^2-51x-53 &= 0\\
(x+1)(2x-53) &= 0\\
x &= -1,\ 53/2
\end{aligned}$$

Checking both answers yields -1 as a solution, but $53/2$ is not a solution as it makes the left hand side take the logarithm of a negative.

(b) Taking the natural logarithm of both sides yields

$$\begin{aligned}
\ln 3^{x-1} &= \ln 4\left(5^{2x+3}\right)\\
\ln 3^{x-1} &= \ln 4+\ln 5^{2x+3}\\
(x-1)\ln 3 &= \ln 4+(2x+3)\ln 5\\
x\ln 3-1\ln 3 &= \ln 4+2x\ln 5+3\ln 5\\
x\ln 3-2x\ln 5 &= \ln 3+\ln 4+3\ln 5\\
x(\ln 3-2\ln 5) &= \ln 3+\ln 4+3\ln 5\\
x &= \frac{\ln 3+\ln 4+3\ln 5}{\ln 3-2\ln 5}
\end{aligned}$$

(c)

$$e^{2x}-5e^{x}-14=0 \Rightarrow (e^x-7)(e^x+2)=0 \Rightarrow e^x=-2,\ 7.$$

Since the exponential e^x is always positive, this means $e^x=7\Rightarrow x=\ln 7$.

[3] 5. (a) Sketch the graph

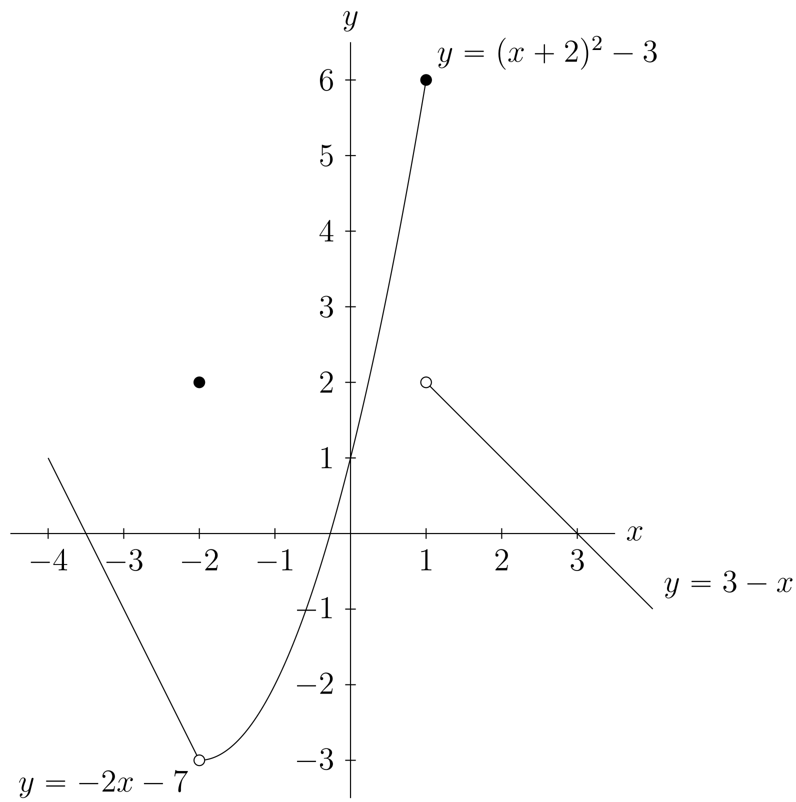
$$f(x) = \begin{cases} -2x - 7 & x < -2 \\ 2 & x = -2 \\ (x+2)^2 - 3 & -2 < x \leqslant 1 \\ 3 - x & x > 1 \end{cases}$$

[4] (b) Find the following limits if they exists. If they don't exist, explain why

$\lim_{x\to -2^-} f(x)$ $\lim_{x\to -2^+} f(x)$ $\lim_{x\to -2} f(x)$ $\lim_{x\to 0} f(x)$ $\lim_{x\to 1^-} f(x)$ $\lim_{x\to 1^+} f(x)$ $\lim_{x\to 1} f(x)$

[3] (c) Is the function continuous at the values $x = -2, x = 0$ and $x = 1$. Explain your answer.

Solution:



(a)

(b) $\lim_{x\to -2^-} f(x) = -3$

$\lim_{x\to -2^+} f(x) = -3$

$\lim_{x\to -2} f(x) = -3$

$\lim_{x\to 0} f(x) = 1$

$\lim_{x\to 1^-} f(x) = 6$

$\lim_{x\to 1^+} f(x) = 2$

$\lim_{x\to 1} f(x)$ does not exist since the left and right hand limits are not the same.

(c) The function is not continuous at $x = -2$ since the function $f(-2) = 2$ is not the same as the limit $\lim_{x\to -2} f(x) = -3$.

The function is continuous at $x = 0$ since the function $f(0) = 1$ exists, the limit $\lim_{x\to 0} f(x) = 1$ exists and they are equal to each other.

The function is not continuous at $x = 1$ since the limit $\lim_{x\to 1} f(x)$ does not exist.

6. Find the following limits if they exist. If they don't exist, determine whether the limit is ∞, $-\infty$ or neither.

[1] (a) $\displaystyle\lim_{x\to 1^+} \frac{x^2+4x}{2x^2+3x+6}$

[3] (b) $\displaystyle\lim_{x\to 1} \frac{x^3-3x^2-4x+6}{x^2-1}$ (Hint: If $x = 1$ makes a polynomial equal to 0 then what must be a factor?)

[4] (c) $\displaystyle\lim_{x\to 3} \frac{x^2+2x-15}{\sqrt{x-2}-\sqrt{4-x}}$

[4] (d) $\displaystyle\lim_{x\to 3^-} \frac{x^2-7x+10}{x^2-6x+9}$

[4] (e) $\displaystyle\lim_{x\to 2} \frac{x^2+2x-8}{|6-3x|}$

[3] (f) $\displaystyle\lim_{x\to 0^+} x\sin\left(\frac{2}{x^2}\right)$

[4] (g) $\displaystyle\lim_{x\to -\infty} \frac{3x+5}{6x+\sqrt{x^2-2x}}$

[3] (h) $\displaystyle\lim_{x\to \infty} \frac{1}{x-\sqrt{x^2+6x}}$

Solution:

(a) $\displaystyle\lim_{x\to 1^+} \frac{x^2+4x}{2x^2+3x+6} = \frac{(1)^2+4(1)}{2(1)^2+3(1)+6} = \frac{5}{11}.$

(b) $\lim_{x\to 1} \dfrac{x^3-3x^2-4x+6}{x^2-1}$ is in 0/0 form. Thus plugging in $x=1$ means both polynomials are zero. Hence $x-1$ factors both the numerator and the denominator.

$$\lim_{x\to 1} \frac{x^3-2x^2-4x+5}{x^2-1} = \lim_{x\to 1} \frac{(x-1)(x^2-2x-6)}{(x-1)(x+1)} = \lim_{x\to 1^+} \frac{x^2-2x-6}{x+1} = -\frac{7}{2}.$$

(c) $\lim_{x\to 3} \dfrac{x^2+2x-15}{\sqrt{x-2}-\sqrt{4-x}}$ is in 0/0 form. Since we have square roots, we won't be able to find the factor of $x-2$ in the denominator until we multiply by the conjugate.

$$\begin{aligned}
\lim_{x\to 3} \frac{x^2+2x-15}{\sqrt{x-2}-\sqrt{4-x}} &= \lim_{x\to 3} \frac{(x+5)(x-3)\left(\sqrt{x-2}+\sqrt{4-x}\right)}{\left(\sqrt{x-2}-\sqrt{4-x}\right)\left(\sqrt{x-2}+\sqrt{4-x}\right)} \\
&= \lim_{x\to 3} \frac{(x+5)(x-3)\left(\sqrt{x-2}+\sqrt{4-x}\right)}{x-2-(4-x)} \\
&= \lim_{x\to 3} \frac{(x+5)(x-3)\left(\sqrt{x-2}+\sqrt{4-x}\right)}{2x-6} \\
&= \lim_{x\to 3} \frac{(x+5)\left(\sqrt{x-2}+\sqrt{4-x}\right)}{2} \\
&= \frac{(8)(2)}{2} \\
&= 8.
\end{aligned}$$

(d)

$$\lim_{x\to 3^-} \frac{x^2-7x+10}{x^2-6x+9}$$

is in $-2/0$ form, Hence the limit goes to either $\pm\infty$.

The numerator is negative when $x=3$ and as $x\to 3^-$ the denominator is positive as the denominator is squared. Hence the limit is $-\infty$.

(e) $\lim_{x\to 2} \frac{x^2+2x-8}{|6-3x|}$ is in 0/0 form. However the $|6-2x|$ is a piecewise function which changes at $x=2$. Thus we must take one-sided limits.

$$\lim_{x\to 2^+} \frac{x^2+2x-8}{|6-3x|} = \lim_{x\to 2^+} \frac{(x+4)(x-2)}{-(6-3x)} = \lim_{x\to 2^+} \frac{x+4}{3} = 2$$

$$\lim_{x\to 2^-} \frac{x^2+2x-8}{|6-3x|} = \lim_{x\to 2^-} \frac{(x+4)(x-2)}{6-3x} = \lim_{x\to 2^-} \frac{x+4}{-3} = -2$$

Since the left and right hand limits are not the same, the limit does not exist, and is neither ∞ nor $-\infty$.

(f) $\lim_{x\to 0^+} x\sin\left(\frac{2}{x^2}\right)$ doesn't follow the limits laws as the limit of $\sin\left(\frac{2}{x^2}\right)$ does not exist. However, we can use the squeeze theorem.

Using that $\sin\left(\frac{2}{x^2}\right)$ is bounded between -1 and 1, we know that

$$-x \leqslant x\sin\left(\frac{2}{x^2}\right) \leqslant x$$

for $x > 0$. Since

$$\lim_{x\to 0^+} -x = \lim_{x\to 0^+} x = 0,$$

by the squeeze theorem we know that $\lim_{x\to 0^+} x\sin\left(\frac{2}{x^2}\right) = 0$.

(g) Factoring out the largest power of x which is either x or $\sqrt{x^2} = |x|$ yields

$$\begin{aligned}
\lim_{x\to -\infty} \frac{3x+5}{6x+\sqrt{x^2-2x}} &= \lim_{x\to -\infty} \frac{x(3+5/x)}{6x+|x|\sqrt{1-2/x}} \\
&= \lim_{x\to -\infty} \frac{x(3+5/x)}{6x-x\sqrt{1-2/x}} \\
&= \lim_{x\to -\infty} \frac{3+5/x}{6-\sqrt{1-2/x}} \\
&= \frac{3}{5}
\end{aligned}$$

(h) The denominator is of $\infty - \infty$ form which is indeterminant. Therefore we want to multiply by the conjugate

$$\begin{aligned}
\lim_{x\to\infty} \frac{1}{x-\sqrt{x^2+6x}} &= \lim_{x\to\infty} \frac{x+\sqrt{x^2+6x}}{(x-\sqrt{x^2+6x})(x+\sqrt{x^2+6x})} \\
&= \lim_{x\to\infty} \frac{x+\sqrt{x^2+6x}}{-6x} \\
&= \lim_{x\to\infty} \frac{x(1+\sqrt{1+6/x})}{-6x} \\
&= \lim_{x\to\infty} \frac{1+\sqrt{1+6/x}}{-6} \\
&= \frac{1+\sqrt{1}}{-6} \\
&= -\frac{1}{3}
\end{aligned}$$

[5] 7. Use limits to calculate a and b such that the following function is continuous for **all real numbers** x. Justify your answer.

$$f(x) = \begin{cases} x^2+ax+b & x<2 \\ 5 & x=2 \\ \ln(x-1)+(b-a)x-9 & x>2 \end{cases}$$

Solution:

The function is continuous for $x<2$ since x^2+ax+b is a polynomial and polynomials are always continuous. The function is continuous for $x<2$ since $\ln(x-1)+(b-a)x-9$ is the addition of a logarithm and a polynomial and both are always continuous as long as we take the logarithm of a positive value. Hence we focus at $x=2$

For the function to be continuous at $x=2$ we must have that

$$\lim_{x\to 2^-} f(x) = f(2) = \lim_{x\to 2^+} f(x)$$

From $\lim_{x\to 2^-} f(x) = f(2)$ we have that

$$
\begin{aligned}
\lim_{x\to 2^-} f(x) &= f(2)\\
\lim_{x\to 3^-} (x^2+ax+b) &= 5\\
4+2a+b &= 5\\
2a+b &= 1.
\end{aligned}
$$

From $\lim_{x\to 2^+} f(x) = f(2)$ we have that

$$
\begin{aligned}
\lim_{x\to 2^+} f(x) &= f(2)\\
\lim_{x\to 2^-} (\ln(x-1)+(b-a)x-9) &= 5\\
\ln 1 + 2(b-a) - 9 &= 5\\
2b-2a &= 14.\\
b &= a+7.
\end{aligned}
$$

Since $b = a+7$. Inserting that into the first equation yields

$$2a + (a+7) = 1 \Rightarrow 3a = -6 \Rightarrow a = -2 \Rightarrow b = 5.$$

Hence $a = -2$ and $b = 5$.

[4] 8. Show that $2x^4 = 4x - 1$ has a solution on the interval $(1, 2)$. Justify your answer.

Solution: Let $f(x) = 2x^4 - 4x + 1$. We are trying to show $f(x) = 0$ has a solution on the interval $(1, 2)$. Since f is the addition of a exponential and a polynomial, f is continuous and therefore the Intermediate Value Theorem applies.

$f(1) = -1 < 0$.

$f(2) = 25 > 0$.

Therefore by the Intermediate Value Theorem there exists a value c between 1 and 2 such that $f(c) = 0$.

This assignment is out of 75 points.