MATH 1500 D01 Winter 2017 Assignment 2

SHOW ALL WORK to get full marks. Leave answers as exact answers. For example, leave it as fractions such as 1/7 as opposed to decimals such as 0.142857 or $\ln 4$ or e^{15}. Word problems should have sentence answers with units. Fractions should be lowest terms.

Calculators are not permitted. Assignments using a calculator will not be graded.

All assignments must be handed in on UMLearn as **one PDF file**. Late assignments will not be accepted. Failure to follow the instructions will result in a mark of 0.

Techniques from this course must be used to solve the questions, not more advanced techniques. For example L'Hopital's Rule for solving limits is not permitted.

This assignment covers sections from 2.7, 2.8, 3.1–3.5, 3.9.

Use the definition of derivative only for Q1. Do not use the definition in any other questions.

[7] 1. Use the definition of the derivative to calculate the derivative of the function $f(x) = \dfrac{1}{\sqrt{5-x^2}}$.

Solution:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{5-(x+h)^2}} - \dfrac{1}{\sqrt{5-x^2}}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{5-x^2}-\sqrt{5-(x+h)^2}}{h\sqrt{5-x^2}\sqrt{5-(x+h)^2}} \\
&= \lim_{h\to 0} \frac{\left(\sqrt{5-x^2}-\sqrt{5-(x+h)^2}\right)\left(\sqrt{5-x^2}+\sqrt{5-(x+h)^2}\right)}{h\sqrt{5-x^2}\sqrt{5-(x+h)^2}\left(\sqrt{5-x^2}+\sqrt{5-(x+h)^2}\right)} \\
&= \lim_{h\to 0} \frac{(5-x^2)-(5-(x+h)^2)}{h\sqrt{5-x^2}\sqrt{5-(x+h)^2}\left(\sqrt{5-x^2}+\sqrt{5-(x+h)^2}\right)} \\
&= \lim_{h\to 0} \frac{2xh+h^2}{h\sqrt{5-x^2}\sqrt{(x+h)^2+5}\left(\sqrt{5-x^2}+\sqrt{(x+h)^2+5}\right)} \\
&= \lim_{h\to 0} \frac{2x+h}{\sqrt{5-x^2}\sqrt{5-(x+h)^2}\left(\sqrt{5-x^2}+\sqrt{5-(x+h)^2}\right)} \\
&= \frac{2x+0}{\sqrt{5-x^2}\sqrt{5-(x+0)^2}\left(\sqrt{5-x^2}+\sqrt{5-(x+0)^2}\right)} \\
&= \frac{x}{(\sqrt{5-x^2})^3}
\end{aligned}
$$

2. Calculate the derivative of the following functions. Do not simplify your answers.

[3] (a) $f(x) = \sqrt[4]{x} + \dfrac{2}{x^{2017}} - \dfrac{2}{\sqrt{x^5}}$

[4] (b) $f(x) = \dfrac{5 - x\sin x}{x + e^x}$

[5] (c) $f(x) = e^{e^{3\cot x}}$

[5] (d) $f(x) = \cos(\sqrt{e^{5x} + \tan x})$

Solution:

(a) $f(x) = x^{1/4} + 2x^{-2017} - 2x^{-5/2}$

Therefore

$$f'(x) = \frac{1}{4}x^{-3/4} - 4034x^{-2018} + 5x^{-7/2}$$

(b)

$$\begin{aligned} f'(x) &= \frac{(5 - x\sin x)'\,(x + e^x) - (5 - x\sin x)(x + e^x)'}{(x + e^x)^2} \\ &= \frac{(-(x\cos x + \sin x))(x + e^x) - (5 - x\sin x)(1 + e^x)}{(x + e^x)^2} \end{aligned}$$

(c)

$$\begin{aligned} f'(x) &= e^{e^{3\cot x}} \cdot \left(e^{3\cot x}\right)' \\ &= e^{e^{3\cot x}} \cdot e^{3\cot x}(3\cot x)' \\ &= e^{e^{3\cot x}} \cdot e^{3\cot x}(-3\csc^2 x) \end{aligned}$$

(d)

$$\begin{aligned} f'(x) &= -\sin(\sqrt{e^{5x} + \tan x})(\sqrt{e^{5x} + \tan x})' \\ &= -\sin(\sqrt{e^{5x} + \tan x})\frac{1}{2}(e^{5x} + \tan x)^{-1/2}(e^{5x} + \tan x)' \\ &= -\frac{1}{2}\sin(\sqrt{e^{5x} + \tan x})(e^{5x} + \tan x)^{-1/2}(5e^{5x} + \sec^2 x) \end{aligned}$$

3. Evaluate the following limits. Show all steps.

[4] (a) $\lim_{x \to 0} \frac{3 \sin 4x}{\sin 7x}$

[4] (b) $\lim_{t \to 3} \frac{4 \sin(t-3)}{(t^2 - 8t + 15)}$

Solution: We will repeatedly use that $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$.

(a)

$$\begin{aligned}
\lim_{x \to 0} \frac{3 \sin 4x}{\sin 7x} &= 3 \lim_{x \to 0} \frac{\sin 4x}{4x} \cdot \frac{7x}{\sin 7x} \cdot \frac{4x}{7x} \\
&= \frac{12}{7} \lim_{x \to 0} \frac{\sin 4x}{4x} \cdot \frac{1}{\lim_{x \to 0} \frac{\sin 7x}{7x}} \\
&= \frac{12}{7} \cdot 1 \cdot \frac{1}{1} \\
&= \frac{12}{7}
\end{aligned}$$

(b) Suppose $\theta = t - 3$ Then $\theta \to 0$ as $t \to 3$. Hence the limit is

$$\begin{aligned}
\lim_{t \to 3} \frac{4 \sin(t-3)}{(t^2 - 8t + 15)} &= \lim_{t \to 3} \frac{2 \sin(t-2)}{(t-3)(t-5)} \\
&= 4 \lim_{t \to 3} \frac{\sin(t-3)}{t-3} \cdot \lim_{t \to 3} \frac{1}{t-5} \\
&= 4 \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \cdot \lim_{t \to 3} \frac{1}{t-5} \\
&= 4 \cdot 1 \cdot \frac{1}{3-5} \\
&= -2
\end{aligned}$$

Note: The substitution could be used to change the entire limit to θ and obtain the new limit

$$\lim_{\theta \to 0} \frac{4 \sin \theta}{\theta^2 - 2\theta}.$$

Afterwards, the steps would follow similarly.

4. If a rock is thrown upwards with a velocity of 20 metres per second, its height, in metres, is given by

$$H(t) = -4.9t^2 + 20t.$$

(Calculus method and not physics methods must be used)

[1] (a) Find the velocity at any time t seconds.

[3] (b) After how many seconds will the rock stop rising and begin to fall?

[4] (c) After what time, and with what velocity will the rock hit the ground?

Solution:

(a) The velocity is just the derivative.

$v(t) = H'(t) = -9.8t + 20.$

(b) This happens when the velocity is 0. Hence

$$0 = -9.8t + 20 \Rightarrow t = \frac{20}{9.8}$$

Therefore the rock will stop rising and being to fall at $\frac{20}{9.8}$ seconds.

(c) The ball will hit the ground when the height is 0. Therefore

$$0 = H(t) = -4.9t^2 + 20t = t(-4.9t + 20) \Rightarrow t = 0,\ t = \frac{20}{4.9}$$

Since $t = 0$ is when the rock is thrown into the ball, it will hit the ground in $\frac{20}{4.9}$ seconds.

Plugging in $t = \frac{20}{4.9}$ into the velocity function yields

$$v\left(\frac{20}{4.9}\right) = -9.8\left(\frac{20}{4.9}\right) + 15 = -40 + 20 = -20.$$

Therefore the velocity when the ball hits the ground will be -20 metres per second (or 15 metres per seconds in the downward direction.)

[5] 5. Calculate an equation of the tangent line to $f(x) = (x^4 + 4x - 5)e^x$ when $x = 0$.

Solution:

Finding the y- coordinate yields $x = 0 \Rightarrow y = f(0) = (0 + 0 - 5)e^0 = -5$.

$f'(x) = (x^4 + 4x - 5)'\,e^x = (x^4 + 4x - 5)(e^x)' = (4x^3 + 4)e^x + (x^4 + 4x - 5)e^x$. Hence $f'(0) = (4(0)^3 + 4)e^0 + (0^4 + 4(0) - 5)e^0 = -1$.

Hence an equation is

$$y + 5 = -1(x - 0) \text{ or } y = -x - 5.$$

[4] 6. If $f(2) = 4$ and $f'(2) = -3$, calculate $g'(2)$ if $g(x) = x^2 f(x)$.

Solution:

$g'(x) = 2xf(x) + x^2 f'(x)$.

Therefore $g'(2) = 2(2)f(2) + (2)^2 f'(2) = 4(4) + 4(-3) = 4$.

[8] 7. Use implicit differentiation to find the equation of the tangent line to the curve

$$x^3 + \sin xy + y^3 = e^{2y} - 1$$

at the point $(1, 0)$.

Solution:

Method 1: Find dy/dx in general

Taking the implicit derivative obtains

$$3x^2 + \cos(xy)(y + xy') + 3y^2y' = 2e^{2y}y'$$

Simplifying:

$$3x^2 + \cos(xy)(y + xy') + 3y^2y' = 2e^{2y}y'$$

Isolating y'

$$3x^2 + y\cos(xy) = -x\cos(xy)y' - 3y^2y' + 2e^{2y}y'$$

Factor y' and divide

$$y' = \frac{3x^2 - +y\cos(xy)}{-x\cos(xy) - 3y^2 + 2e^{2y}}$$

The slope at the point (1, 0) is therefore

$$m = y'|_{(x,y)=(1,0)} = \frac{3(1)^2 + 0\cos(0)}{-\cos(0) - 0 + 2e^0} = \frac{3}{1} = 3$$

Hence the equation of the tangent line is $y = 3(x - 1)$.

Method 2: Just find y' at the given point

Taking the implicit derivative obtains

$$3x^2 + \cos(xy)(y + xy') + 3y^2y' = 2e^{2y}y'$$

Plugging in $(x, y) = (1, 0)$ and letting $m = y'|_{(x,y)=(1,0)}$ yields

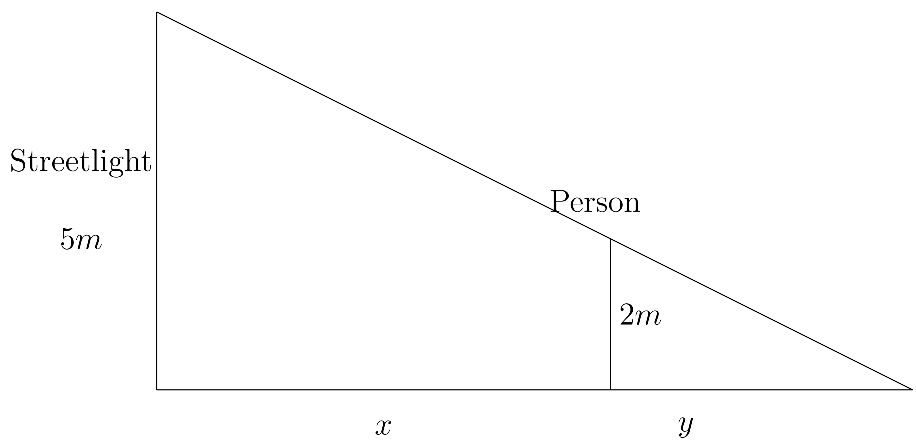
$$3 + \cos(0)(0 + m) + 0 = 2e^0m \Rightarrow 3 + m = 2m \Rightarrow 3 = m.$$

Hence the equation of the tangent line is $y = 3(x - 1)$.

[9] 8. A streetlight, 5 metres tall, shines on the ground. A person 2 metres tall walks towards the streetlight at a speed of 4/5 metres per second. How fast is the length of their shadow changing when they are 4 meters from the streetlight

Solution:

A picture is below:



Let y be the length of the shadow at any time and x be the distance between the person and the streetlight at any time.

$dx/dt = -4/5$ m/s and we want to compute dy/dt when $x = 4$ m.

To obtain an equation, we can use the two similar triangles.

$$\frac{y}{2} = \frac{x+y}{5} \Rightarrow 5y = 2x + 2y \Rightarrow y = \frac{2x}{3}$$

Taking the derivative with respect to time yields:

$$\frac{dy}{dt} = \frac{2}{3}\frac{dx}{dt}$$

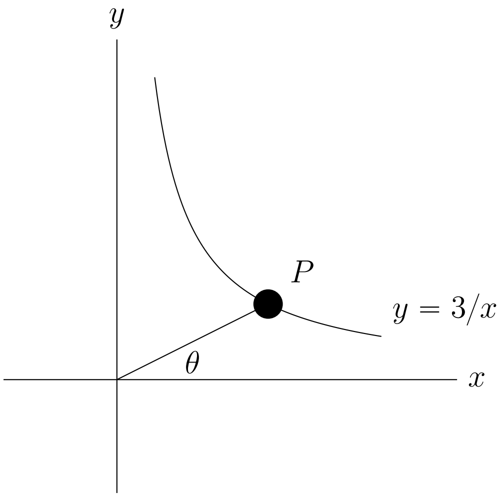
At the given rate for x this yields

$$\frac{dy}{dt} = \frac{2}{3} \cdot \frac{-4}{5} = -\frac{8}{15}.$$

Hence the length of the shadow is decreasing at a rate of 8/15 m/s.

(Note: In this case, the 4 metres was not necessary. The rate the shadow is changing is solely dependent on the rate the person is walking, not where they are.)

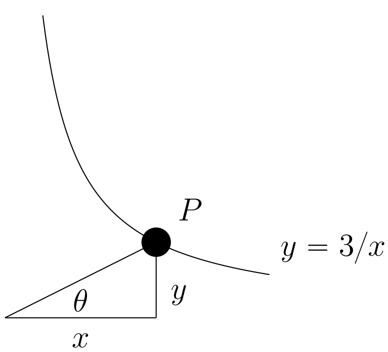
[9] 9. A particle P is moving along the curve $y = \frac{3}{x}$ so that its x coordinate (in meters) is increasing at a rate of 4 m/s. The line segment between the origin (0, 0) and P forms an angle θ between the line segment and the positive x-axis. See the figure below



Compute the rate of change of θ with respect to time when the particle passes through $x = \sqrt{3}$. (Hint: You may use $\sec^2\theta = 1 + \tan^2\theta$ at some point, although it is not necessary.)

Solution:

Completing the picture



We know $\frac{dx}{dt} = 4m/s$ and we want to find $\frac{d\theta}{dt}$ when $x = \sqrt{3}m$

Thus we have the equation $\tan\theta = \frac{y}{x} = \frac{3}{x^2}$.

Taking the derivative yields

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{-6}{x^3}\frac{dx}{dt}$$

At $x = \sqrt{3}$ we have that $\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$ and thus $\sec^2\theta = \left(\frac{1}{\cos\pi/4}\right)^2 = \frac{1}{1/2} = 2$.

Note that we could also use $\sec^2\theta = 1 + \tan^2\theta = 1 + 1^2 = 2$.

Hence at the given point we have

$$2 \cdot \frac{d\theta}{dt} = \frac{-6}{3\sqrt{3}}(4) \Rightarrow \frac{d\theta}{dt} = -\frac{4}{\sqrt{3}}.$$

Hence at $x = 2m$ we have that the angle is decreasing at a rate of $4/\sqrt{3}$ radians per second.

This assignment is out of 75 points.