MATH 1500 D01 Winter 2017 Assignment 3

SHOW ALL WORK to get full marks. Leave answers as exact answers. For example, leave it as fractions such as 1/7 as opposed to decimals such as 0.142857 or $\ln 4$ or e^{15}. Word problems should have sentence answers with units. Fractions should be lowest terms.

Calculators are not permitted. Assignments using a calculator will not be graded.

All assignments must be handed in on UMLearn as **one PDF file**. Late assignments will not be accepted. Failure to follow the instructions will result in a mark of 0.

Techniques from this course must be used to solve the questions, not more advanced techniques. For example L'Hopital's Rule for solving limits is not permitted.

This assignment covers sections from 3.6, 4.1-4.5

1. Compute the derivative of the following functions:

[5] (a) $f(x) = 4^{3^{6\cot x}}$

Solution:

$$\begin{aligned} f'(x) &= 4^{3^{6\cot x}} \cdot \ln 4 (3^{6\cot x})' \\ &= 4^{3^{6\cot x}} \cdot \ln 4 \cdot 3^{6\cot x} \cdot \ln 3 (6\cot x)' \\ &= 4^{3^{6\cot x}} \cdot \ln 4 \cdot 3^{6\cot x} \cdot \ln 3(-6\csc^2 x) \end{aligned}$$

[5] (b) $f(x) = \ln|\csc(\pi^2 - t^4)|$

Solution:

$$\begin{aligned} f'(x) &= \frac{1}{\csc(\pi^2 - t^4)} \cdot (\csc(\pi^2 - t^4))' \\ &= \frac{1}{\csc(\pi^2 - t^4)}(-\csc(\pi^2 - t^4)\cot(\pi^2 - t^4)) \cdot (\pi^2 - t^4)' \\ &= \frac{1}{\csc(\pi^2 - t^4)}(-\csc(\pi^2 - t^4)\cot(\pi^2 - t^4)) \cdot (-4t^3) \end{aligned}$$

[5] (c) $f(x) = \log_4(\cos 3x + \ln x)$

Solution:

$$\begin{aligned} f'(x) &= \frac{1}{\ln 4(\cos 3x + \ln x)} \cdot (\cos 3x + \ln x)' \\ &= \frac{1}{\ln 4(\cos 3x + \ln x)} \cdot (-3\sin 3x + 1/x) \end{aligned}$$

[6] (d) $f(x) = x^3 + (\ln x)^{x^3}$

Solution:
Starting with $y = (\ln x)^{x^3}$
Taking logarithms yields

$$\ln y = x^3 \ln(\ln x)$$

Taking the derivative

$$\frac{1}{y}\frac{dy}{dx} = 3x^2 \ln\ln x + \frac{x^3}{x \ln x}$$

Thus

$$\frac{dy}{dx} = y\left(3x^2 \ln\ln x + \frac{x^3}{x \ln x}\right) = (\ln x)^{x^3}\left(3x^2 \ln\ln x + \frac{x^3}{x \ln x}\right).$$

Thus the overall derivative is

$$f'(x) = 3x^2 + (\ln x)^{x^3}\left(3x^2 \ln\ln x + \frac{x^3}{x \ln x}\right).$$

[7] 2. Use logarithmic differentation to compute the derivative of

$$f(x) = \frac{\sqrt[3]{4x+3}\, e^{-6x}(x-2)^9}{x^3}.$$

Solution:
Taking logarithms

$$\ln f(x) = \ln \frac{\sqrt[3]{4x+3}e^{-6x}(x-5)^7}{x^3} = \frac{1}{3}\ln(4x+3) - 6x + 9\ln(x-2) - 3\ln x$$

Taking the derivative leads to

$$\frac{f'(x)}{f(x)} = \frac{1}{3(4x+3)} - 6 + \frac{9}{x-2} - \frac{3}{x}$$

Hence

$$\begin{aligned} f'(x) &= f(x)\left(\frac{1}{3(4x+3)} - 6 + \frac{9}{x-2} - \frac{3}{x}\right) \\ &= \frac{\sqrt[3]{4x+3}\, e^{-6x}(x-2)^9}{x^3}\left(\frac{1}{3(4x+3)} - 6 + \frac{9}{x-2} - \frac{3}{x}\right) \end{aligned}$$

3. Compute the critical numbers of the following functions. Show complete reasoning.

[3] (a) $f(x) = x^3 - 4x^2 - 3x - 7$

Solution: Critical numbers exist when $f'(x) = 0$ or is undefined. Taking the derivative

$$f'(x) = 3x^2 - 8x - 3$$

which is a polynomial and thus always exists. Setting equal to 0 yields

$$0 = f'(x) = 3x^2 - 8x - 3 = (3x+1)(x-3) \Rightarrow x = -1/3, 3.$$

Hence the critical numbers are $-1/3$ and 3.

[3] (b) $f(x) = |7x - 4|$

Solution: Critical numbers exist when $f'(x) = 0$ or is undefined. However,

$$f(x) = \begin{cases} -(7x-4) & x < 4/7 \\ 7x-4 & x \geqslant 4/7 \end{cases} \Rightarrow f'(x) = \begin{cases} -7 & x < 4/7 \\ 7 & x > 4/7 \end{cases}$$

but $f'(4/7)$ is undefined. $f'(x)$ is never zero and thus the only critical number is $4/7$.

[6] 4. Compute the absolute maximum and minimum values of $f(x) = \sqrt[5]{x}(6 - x)$ on the interval $[-1, 32]$. Explain all reasoning.

Solution:

Since f is the product of a radical function with domain of all real numbers and a polynomial, it is continuous for all real numbers and thus on $[-1, 32]$. Hence the extreme value theorem applies. Taking the derivative.

$$f(x) = 6x^{1/5} - x^{6/5} \Rightarrow f'(x) = \frac{6}{5}x^{-4/5} - \frac{6}{5}x^{1/5} = \frac{6 - 6x}{5x^{4/5}}.$$

Hence $f'(x)$ is undefined at $x = 0$ and is zero when $6 - 6x = 0 \Rightarrow x = 1$.

Test the critical numbers and endpoints, $x = -1$, 0, 1, 32 leads to

$$\begin{aligned} f(-1) &= \sqrt[5]{-1}(6 - (-1)) = -7 \\ f(0) &= \sqrt[5]{0}(6 - (0)) = 0 \\ f(1) &= \sqrt[5]{1}(6 - (1)) = 5 \\ f(32) &= \sqrt[5]{32}(6 - (32)) = -52 \end{aligned}$$

Hence the absolute maximum is 5 and the absolute minimum is -52.

[6] 5. Verify that the following function satisfies the conditions of the Mean Value Theorem on the interval $[-3, 0]$, and then compute c which satisfies the conclusion of the Mean Value Theorem.

$$f(x) = \frac{x+6}{x+4}$$

Solution:

$$f(x) = \frac{x+6}{x+4} \Rightarrow f'(x) = \frac{1(x+4) - 1(x+6)}{(x+4)^2} = \frac{-2}{(x+4)^2}$$

Since f is a rational function, it is continuous on its domain, which is all real numbers except for -4. Since -4 is not in the interval $[-3,\ 0]$ the function is continuous.

Since f' exists for all x except -4 as well, f is differentiable on $(-3,\ 0)$.

The conclusion of the Mean Value Theorem is that there is some c in the interval $(-3,\ 0)$ such that

$$\begin{aligned} f'(c) &= \frac{f(b)-f(a)}{b-a} = \frac{f(0)-f(-3)}{0-(-3)} \\ &\Rightarrow \frac{-2}{(c+4)^2} = \frac{(3/2)-3}{3} = -\frac{1}{2} \\ &\Rightarrow (c+4)^2 = 4 \\ &\Rightarrow c+4 = \pm 2 \\ &\Rightarrow c = -6,\ -2 \end{aligned}$$

Since -6 is not in the interval, the only value of c is -2.

[14] 6. Let

$$f(x) = \frac{3x^2-2}{x^3} \qquad f'(x) = \frac{-3(x^2-2)}{x^4} \qquad f''(x) = \frac{6(x^2-4)}{x^5}.$$

Determine the following. Show all work.

(a) Domain of f:

(b) Symmetry of f:

(c) Equation(s) of any vertical asymptotes:

(d) Equation(s) of any horizontal asymptotes:

(e) Coordinates of any critical point(s):

(f) Open intervals where f is increasing:

(g) Open intervals where f is decreasing:

(h) Coordinates of any local maxima:

(i) Coordinates of any local minima:

(j) Open intervals where f is concave up:

(k) Open intervals where f is concave down:

(l) Coordinates of any inflection point(s):

(m) Sketch the graph of f, labelling all points of interest.

Solution:

(a) x can be anything but 0 as it makes the denominator equal to 0. Hence the domain is $(-\infty, 0) \cup (0, \infty)$

(b) Since $f(-x) = \dfrac{3(-x)^2 - 2}{(-x)^3} = -\dfrac{3x^2 - 2}{x^3} = -f(x)$, then function is odd.

(c) Since the function is rational, it is continuous on the domain and thus the only place where a V.A. can exist is at 0. Testing 0 leads to $\lim_{x\to 0} \dfrac{3x^2 - 2}{x^3}$ which is a $-2/0$ form. Therefore the limit goes to either $\pm\infty$. Hence $x = 0$ is a vertical asymptote.

(d) Testing the limit as $x \to \pm\infty$ leads to

$$\lim_{x\to\pm\infty} \frac{3x^2 - 2}{x^3} = \lim_{x\to\pm\infty} \left(\frac{3}{x} - \frac{2}{x^3}\right) = 0.$$

Thus the horizontal asymptote is $y = 0$.

(e) $f'(x)$ is defined except at $x = 0$. (which is outside the domain) Testing where the derivative is 0 leads to $x^2 - 2 = 0 \Rightarrow x = \pm\sqrt{2}$. If $x = \sqrt{2} \Rightarrow y = \sqrt{2}$ and $x = -\sqrt{2} \Rightarrow y = -\sqrt{2}$. Thus the critical points are $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$.

(f) For the next four parts we use the number line for

f'	$-$		$+$		$+$		$-$
		$-\sqrt{2}$		0		$\sqrt{2}$	
f	↘		↗		↗		↘

Since the function is increasing when the derivative is positive leads to the intervals $(-\sqrt{2}, 0)$ and $(0, \sqrt{2})$

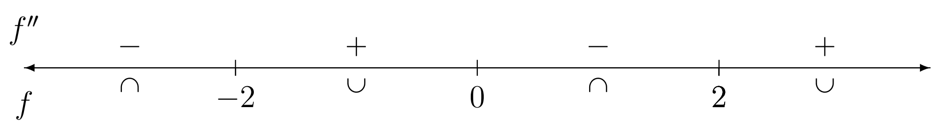
(g) Since the function is decreasing when the derivative is negative leads to the intervals $(-\infty, -\sqrt{2})$ and $(\sqrt{2}, \infty)$

(h) $(\sqrt{2},\ \sqrt{2})$

(i) $(-\sqrt{2},\ -\sqrt{2})$

(j) $f''(x)$ is defined except at $x = 0$. (which is outside the domain) Testing where the derivative is 0 leads to $x^2 - 4 = 0 \Rightarrow x = \pm 2$. If $x = 2 \Rightarrow y = \frac{5}{4}$ and $x = -2 \Rightarrow y = -\frac{5}{4}$. Thus the possible inflection points are $\left(2,\ \frac{5}{4}\right)$ and $\left(-2,\ -\frac{5}{4}\right)$.

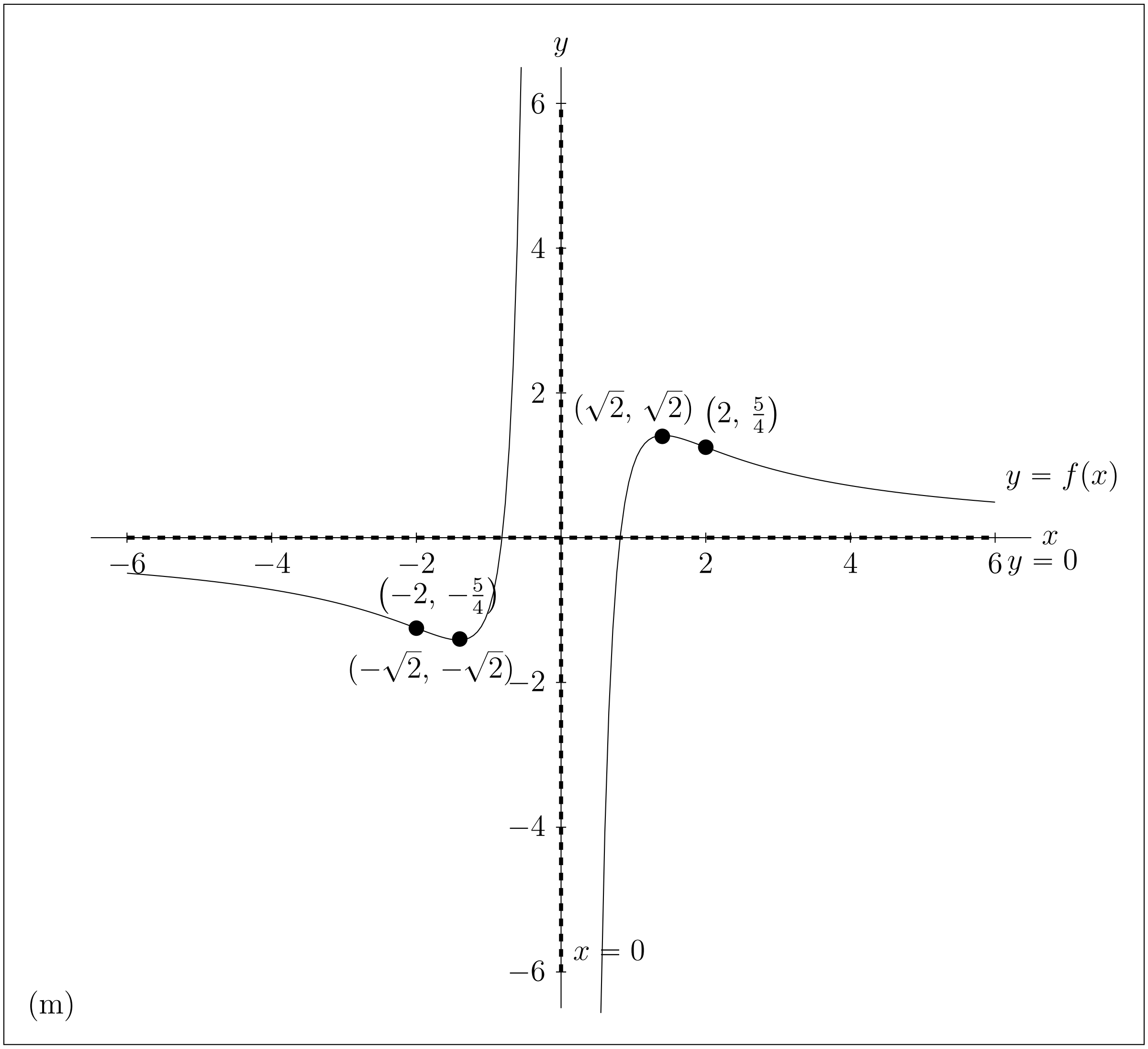
For the concavity we use the number line for



Since the function is concave up when the second derivative is positive leads to the intervals $(-2,\ 0)$ and $(2,\ \infty)$

(k) Since the function is concave down when the second derivative is negative leads to the intervals $(-\infty,\ -2)$ and $(0,\ 2)$

(l) $\left(2,\ \frac{5}{4}\right)$ and $\left(-2,\ -\frac{5}{4}\right)$.



[15] 7. Let

$$f(x) = \frac{x^4}{8(x^2-4)} \qquad f'(x) = \frac{x^3(x^2-8)}{4(x^2-4)^2} \qquad f''(x) = \frac{x^2((x^2-6)^2+60)}{4(x^2-4)^3}$$

Determine the following. Show all work.

(a) Domain of f:

(b) Symmetry of f:

(c) Equation(s) of any vertical asymptotes:

(d) Equation(s) of any horizontal asymptotes:

(e) Coordinates of any critical point(s):

(f) Open intervals where f is increasing:

(g) Open intervals where f is decreasing:

(h) Coordinates of any local maxima:

(i) Coordinates of any local minima:

(j) Open intervals where f is concave up:

(k) Open intervals where f is concave down:

(l) Coordinates of any inflection point(s):

(m) Sketch the graph of f, labelling all points of interest.

Solution:

(a) x can be anything but ± 2 as it makes the denominator equal to 0. Hence the domain is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

(b) Since $f(-x) = \dfrac{(-x)^4}{(-x)^2 - 4} = \dfrac{x^4}{x^2 - 4} = f(x)$, the function is even.

(c) Since the function is rational, it is continuous on the domain and thus the only place where a V.A. can exist is at ± 2. Testing -2 leads to $\lim\limits_{x \to -2} \dfrac{x^4}{x^2 - 4}$ which is a 16/0 form. Therefore the limit goes to either $\pm\infty$. Hence $x = -2$ is a vertical asymptote. $\lim\limits_{x \to 2} \dfrac{x^4}{x^2 - 4}$ which is a 16/0 form. Therefore the limit goes to either $\pm\infty$. Hence $x = 2$ is a vertical asymptote.

(d) Testing the limit as $x \to \pm\infty$ leads to

$$\lim_{x \to \pm\infty} \frac{x^4}{x^2 - 4} = \lim_{x \to \pm\infty} \frac{x^2}{1 - 4/x^2} = \infty.$$

Thus there is no horizontal asymptote.

(e) $f'(x)$ is defined except at $x = \pm 2$. (which is outside the domain) Testing where the derivative is 0 leads to $x^2 = 0 \Rightarrow x = 0$ and $x^2 - 8 = 0 \Rightarrow x = \pm 2\sqrt{2}$. If $x = 2\sqrt{2} \Rightarrow y = 2$ and $x = -2\sqrt{2} \Rightarrow y = 2$. Thus the critical points are $(0, 0)$, $(2\sqrt{2}, 2)$ and $(-2\sqrt{2}, 2)$.

(f) For the next four parts we use the number line for

		$-2\sqrt{2}$		-2		0		2		$2\sqrt{2}$	
f'	$-$		$+$		$+$		$-$		$-$		$+$
f	$\searrow$		$\nearrow$		$\nearrow$		$\searrow$		$\searrow$		$\nearrow$

Since the function is increasing when the derivative is positive leads to the intervals $(-2\sqrt{3}, -2)$, $(-2, 0)$ and $(2\sqrt{3}, \infty)$

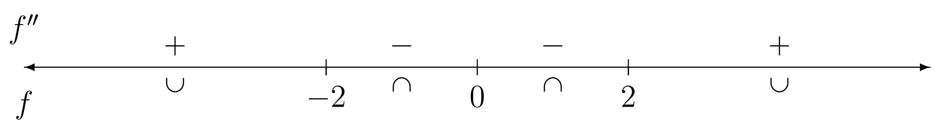
(g) Since the function is decreasing when the derivative is negative leads to the intervals $(-\infty,\ -2\sqrt{2})$, $(0,\ 2)$ and $(2,\ 2\sqrt{3})$

(h) $(0,\ 0)$

(i) $(-2\sqrt{2},\ 2)$, $(2\sqrt{2},\ 2)$

(j) $f''(x)$ is defined except at $x = \pm 2$. (which is outside the domain) Testing where the derivative is 0 leads to $x = 0$ or $(x^2-6)^2+60 = 0 \Rightarrow x^2-6 = \pm\sqrt{-60}$ which is undefined. If $x = 0 \Rightarrow y = 0$. Thus the possible inflection points is $(0, 0)$.

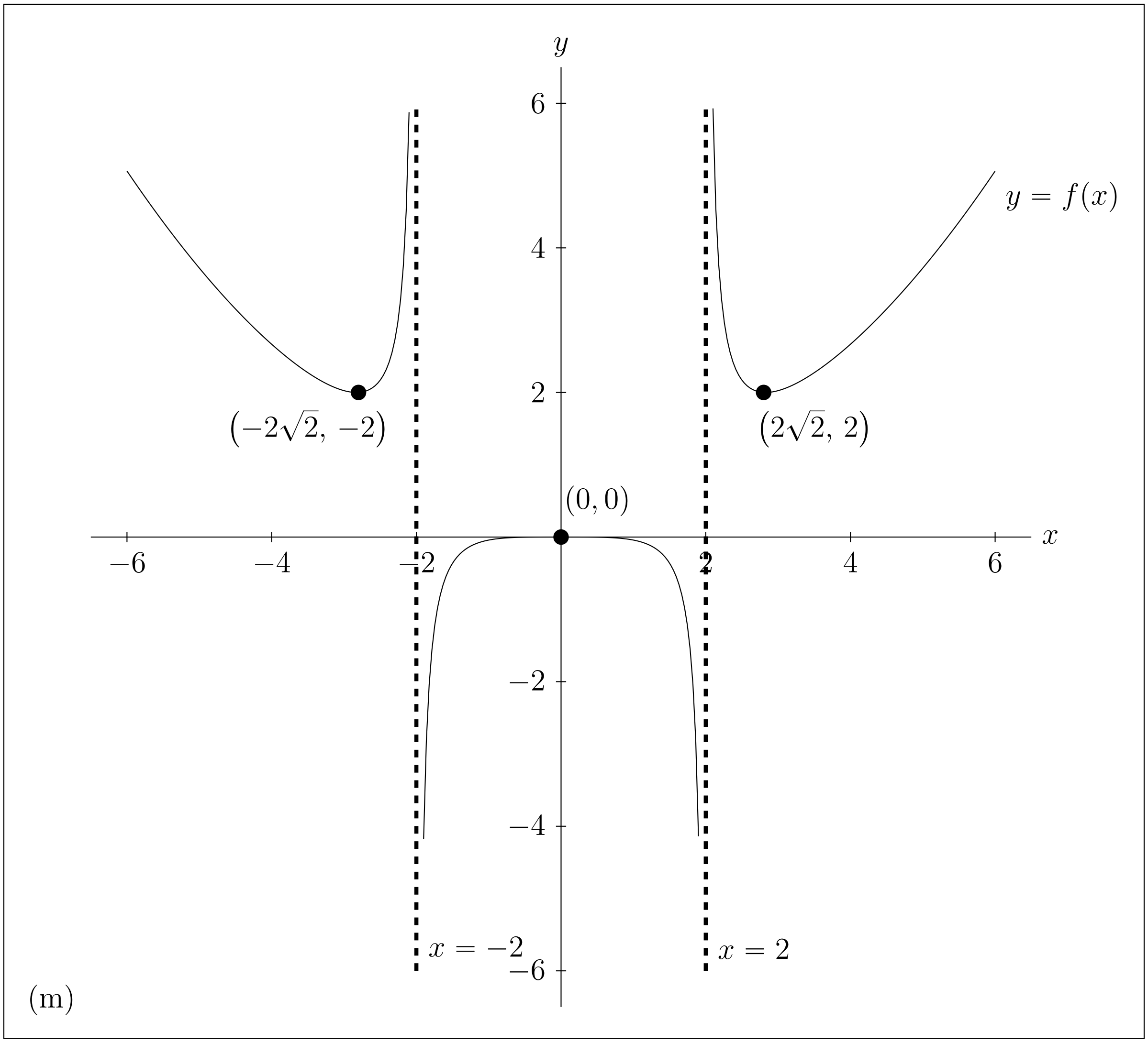
For the concavity we use the number line for



Since the function is concave up when the second derivative is positive leads to the intervals $(-\infty,\ -2)$ and $(2,\ \infty)$

(k) Since the function is concave down when the second derivative is negative leads to the intervals $(0,\ 0)$

(l) Since the concavity did not change at $x = 0$, $(0,\ 0)$ is not a point of inflection. Therefore there is no point of inflection.



This assignment is out of 75 points.