MATH 1500 D01 Winter 2017 Assignment 4

SHOW ALL WORK to get full marks. Leave answers as exact answers. For example, leave it as fractions such as 1/7 as opposed to decimals such as 0.142857 or leave it as $\ln 4$ or e^{15}. Word problems should have sentence answers with units. Fractions should be lowest terms.

Calculators are not permitted. Assignments using a calculator will not be graded.

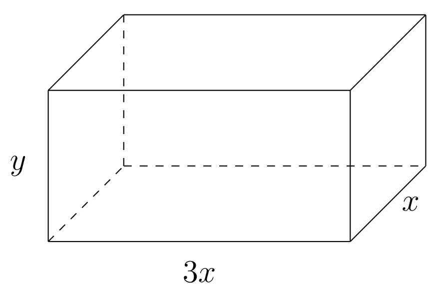
All assignments must be handed in on UMLearn as **one PDF file**. Late assignments will not be accepted. Failure to follow the instructions will result in a mark of 0.

Techniques from this course must be used to solve the questions, not more advanced techniques. For example L'Hopital's Rule for solving limits is not permitted.

The assignment covers sections 4.7, 4.9, 5.1–5.4 in the textbook.

[9] 1. A rectangular storage container is to hold 12 cubic metres. Its length is to be three as long as the width. The top costs \$5 per square metre and the bottom and sides each cost \$4 per square metre. Determine the dimensions which minimize the cost of creating the container. Justify your answer.

Solution:



Let x be the width of the base of the box. Therefore the length is $3x$. Let y be the height of the box.

The top has area $3x^2$ square metres and therefore the cost is $15x^2$ dollars. The bottom and sides have area $3x^2 + 3xy + xy + 3xy + xy = 3x^2 + 8xy$ square metres and therefore the cost is $12x^2 + 32xy$ dollars.

Hence the cost of the box is $27x^2 + 32xy$ dollars.

Since the box is to hold 12 cubic metres, we know $3x^2y = 12 \Rightarrow y = \dfrac{4}{x^2}$

Therefore the cost is $C(x) = 27x^2 + 128x^{-1}$ with domain $(0, \infty)$

The derivative is $C'(x) = 54x - 128x^{-2}$. which is defined on $(0, \infty)$ and is zero when

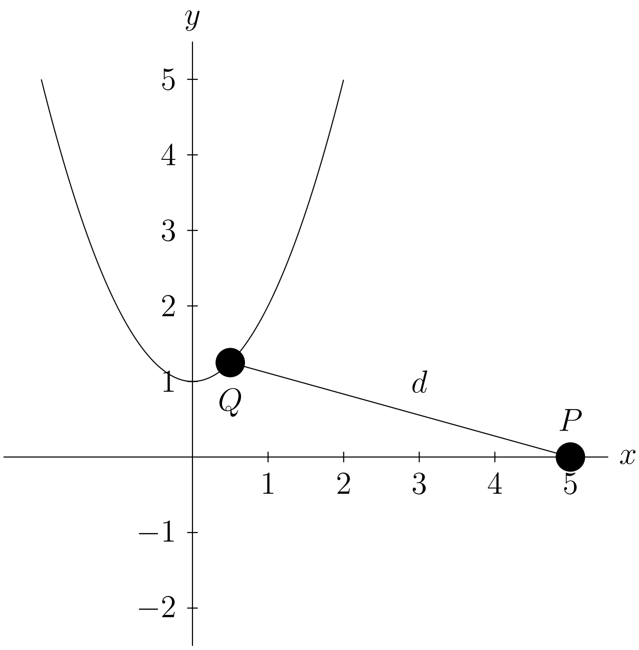
$$54x = 128x^{-2} \Rightarrow 54x^3 = 128 \Rightarrow x^3 = \frac{64}{27} \Rightarrow x = \frac{4}{3}.$$

Using the second derivative test, $C''(x) = 54 + 256x^{-3}$ which is always positive since $x > 0$, we know that $\frac{4}{3}$ yields a local minimum. Since it is the only critical number, it is therefore a absolute minimum.

Thus the width is $\frac{4}{3}$ metres, the length is 4 metres and the height is $\frac{9}{4}$ metres.

[8] 2. Determine the point on the parabola $y = x^2 + 1$ which is closest to the point $P(5, 0)$. Justify your answer.

Solution:



Let D be the square of the distance from the point $P(5, 0)$ to the point $Q(x, y)$ Hence

$$D = (x-5)^2 + y^2 = (x-5)^2 + (x^2+1)^2 = x^4 + 3x^2 - 10x + 26$$

which has domain all real numbers.

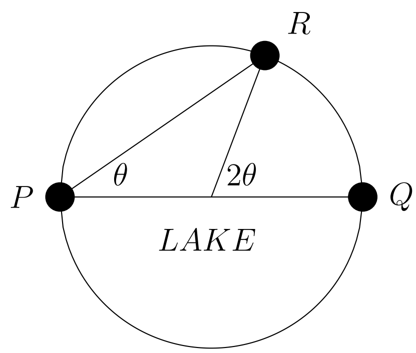
$D'(x) = 4x^3 + 6x - 10$ is always defined and is 0 when $x = 1$.

(Note: $4x^3 + 6x - 10 = (x-1)(4x^2 + 4x + 10)$ for which $4x^2 + 4x + 10 = 0$ has no solution.)

Using the second derivative test, $D''(x) = 8x^2+6$ which is positive at $x = 1$. Therefore $x = 1$ yields a local minimum. Since it is the only critical number, it is therefore a absolute minimum.

Hence the point on the parabola closest to $(5, 0)$ is $(1, 2)$.

[9] 3. You are at point P on the side of a circular lake of radius 20 metres. You need to get to the point Q directly across the lake. You can either walk clockwise around the lake, swim straight across, or swim at an angle θ to a point R on the shore and finish the trip by walking. You walk 2 m/s and swim 1 m/s. Which path minimizes the amount of time to get from P to Q? See picture below. Justify your answer.



Solution:

Let θ be the angle from the picture. The length of the path swimming is $40\cos\theta$ and the length walking is 40θ. Hence the amount of time to get across to point Q is $T(\theta) = \dfrac{40\cos\theta}{1} + \dfrac{40\theta}{2}$. The domain is $[0, \pi/2]$ where 0 is swimming straight across and $\pi/2$ is walking around the lake.

$T'(\theta) = -40\sin\theta + 20$ which is defined for all θ in the domain. Solving where the derivative is zero leads to

$$0 = -40\sin\theta + 20 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}.$$

$T''(\theta) = -40\cos\theta$ which is negative at $\pi/6$. Therefore the critical point leads to a maximum and not a minimum. Hence the minimum must occur at an endpoint

Testing the endpoints leads to

$$\begin{aligned} T(0) &= 40\cos 0 + 20(0) = 40 \\ T(\pi/2) &= 40\cos \pi/2 + 20(\pi/2) = 10\pi \approx 31.4 \end{aligned}$$

Therefore the minimum happens at $\pi/2$ and thus walking around is the fastest.

[7] 4. On planet Millan, the acceleration due to gravity is -16 metres per second squared. A rock is thrown from a height of 12 metres above the surface with an initial upward speed of of 4 metres per second. Assume no other forces acting on the ball except for gravity. Answer the following questions. You are not permitted to use any formulas from physics.

(a) Determine the maximum height that the ball will reach.

(b) Determine the time it takes for the ball to return to the surface.

Solution:

We start by finding the height function

$a(t) = -16 \Rightarrow v(t) = -16t + C$ Since the initial velocity is 4 metres per second which implies $4 = v(0) = 0 + C = C$ Hence $v(t) = -16t + 4$.

$h(t) = -8t^2 + 4t + D$. Since the initial height is 12 metres we have

$$12 = 0 + D = D \Rightarrow h(t) = -8t^2 + 4t + 12.$$

(a) The maximum height is when $v(t) = 0$ and is therefore when $-16t + 4 = 0 \Rightarrow t = 1/4$.

$h(1/4) = -8(1/4)^2 + 4(1/4) + 12 = 25/2$.

Thus the maximum height is 25/2 metres.

(b) The ball returns to the surface when $h(t) = 0$ and $t > 0$.

$$0 = -8t^2 + 4t + 12 \Rightarrow 0 = 2t^2 - t - 3 = (2t - 3)(t + 1) \Rightarrow t = -1,\ 3/2.$$

Therefore the ball returns to the surface in 1.5 seconds.

[7] 5. If $f''(x) = 3\cos x + 24x^2 + 2e^x, f(0) = 3,\ f(\pi) = 0$. Determine the function $f(x)$.

Solution:

$f''(x) = 3\cos x + 24x^2 + 2e^x \Rightarrow f'(x) = 3\sin x + 8x^3 + 2e^x + C \Rightarrow f(x) = -3\cos x + 2x^4 + 2e^x + Cx + D$

From $f(0) = 3$ we have

$$3 = f(0) = -3\cos 0 + 0 + 2e^0 + C(0) + D = -1 + D \Rightarrow D = 4.$$

Hence $f(x) = -3\cos x + 2x^4 + 2e^x + Cx + 4$

From $f(\pi) = 0$ we have

$$0 = f(\pi) = -3\cos\pi + 2\pi^4 + 2e^\pi + C(\pi) + 4 = 2\pi^4 + 2e^\pi + C(\pi) + 7 \Rightarrow C = -\frac{2\pi^4 + 2e^\pi + 7}{\pi}.$$

Therefore $f(x) = -3\cos x + 2x^4 + 2e^x - \left(\frac{2\pi^4 + 2e^\pi + 7}{\pi}\right)x + 4$

[4] 6. Evaluate the following integral by interpreting it in terms of areas. Do not use the fundamental theorem of calculus.

$$\int_0^5 \left(5 - \sqrt{25 - x^2}\right)\, dx$$

Solution:

The integral is the area under the rectangle formed from $y = 5$ from 0 to 5 subtracted by the area under the quarter-circle $y = \sqrt{25 - x^2}$ from 0 to 5

The area of the rectangle is length times width which is $5 \cdot 5 = 25$.

The area of the quarter circle is a quarter of the area of the circle with radius 5 which is $\frac{1}{4}(5)^2\pi = \frac{25\pi}{4}$.

Therefore the integral is $25 - \frac{25\pi}{4}$.

7. Suppose $g(x) = \int_0^x f(t)\,dt$ defined on $[0,\ 10]$ where f is the function sketched below.

[5] (a) Compute $g(0)$, $g(2)$, $g(4)$, $g(6)$ and $g(8)$.

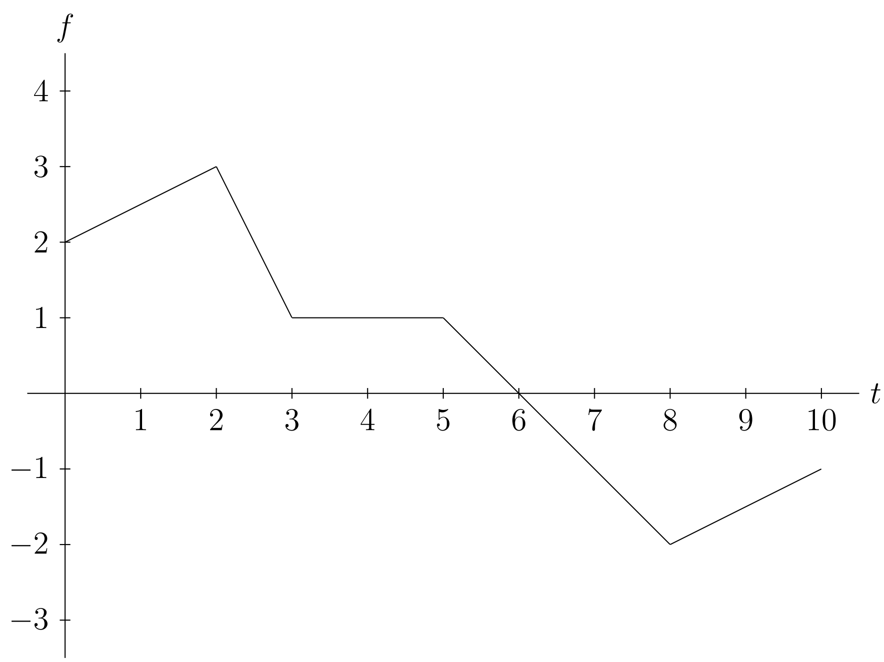
Solution:
$g(x)$ is the area under the curve from 0 to x. Therefore
$g(0) = 0$.
$g(2) = 5$
$g(4)$ is the previous area plus the area from 2 to 4 which is $5 + 3 = 8$
$g(6)$ is the previous area plus the area from 4 to 6 which is $8 + 1.5 = 9.5$
$g(8)$ is the previous area plus the net area from 6 to 8 which is $9.5 - 2 = 7.5$

[2] (b) Determine on which interval that g is increasing. Explain your answer.

Solution:
Since $g(x)$ is increasing while the function is adding positive net area, the function is increasing on $(0, 6)$.

[2] (c) What is the maximum value of g. Explain your answer.

Solution:
The maximum value is $g(6)$ since g is increasing on $(0,\ 6)$ and decreasing afterwards. Therefore the maximum value is 9.5.



8. Compute the derivatives of

[3] (a) $f(x) = \int_2^{\sec x} \sqrt{1 + t^{2017}}\, dt.$

Solution:

$$f'(x) = \sqrt{1 + (\sec x)^{2017}}(\sec x)' = \sqrt{1 + (\tan x)^{2017}}(\sec x \tan x)$$

[4] (b) $f(x) = \int_{x^2}^{x^3} \cos(t^2)\, dt.$

Solution:

$$f(x) = \int_0^{x^3} \cos(t^2)\, dt - \int_0^{x^2} \cos(t^2)\, dt$$

$$f'(x) = \cos((x^3)^2)(x^3)' - \cos((x^2)^2)(x^2)' = 3x^2 \cos(x^6) - 2x \cos(x^4)$$

9. Evaluate the integrals

[4] (a) $\displaystyle\int \left(x^e + \frac{(3x-2)^2}{x}\right) dx.$

Solution:
Simplifying the integral leads to
$\displaystyle\int \left(x^e + 9x - 12 + \frac{4}{x}\right) dx$. Thus the indefinite integral is

$$\frac{x^{e+1}}{e+1} + \frac{9}{2}x^2 - 12x + 4\ln|x| + C.$$

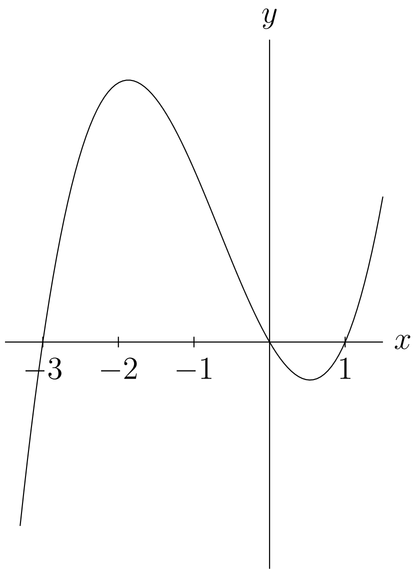
[4] (b) $\displaystyle\int_{\pi/4}^{\pi/2} \csc^2 x \, dx.$

Solution:

$$\begin{aligned}\int_{\pi/4}^{\pi/2} \csc^2 x \, dx &= -\cot x|_{\pi/4}^{\pi/2} \\ &= -\cot(\pi/2) + \cot(\pi/4) \\ &= -0 + 1 \\ &= 1.\end{aligned}$$

[7] 10. Compute the area bounded between the curve $y = 4x^3 + 8x^2 - 12x$ and the x–axis. Include a rough sketch of the region.

Solution:



Computing the x intercepts

$$0 = 4x^3 + 8x^2 - 12x = 4x(x^2 + 2x - 3) = 4x(x+3)(x-1) \Rightarrow x = -3,\ 0,\ 1.$$

From the picture (or testing value in each interval) we can see that the function is positive between -3 and 0 and is negative between 0 and 1. Thus the total area is

$$\int_{-3}^{0} f(x)\,dx - \int_{0}^{1} f(x)\,dx$$

$$\begin{aligned}
\int_{-3}^{0} (4x^3 + 8x^2 - 12x)\,dx &= x^4 + \frac{8}{3}x^3 - 6x^2 \Big|_{-3}^{0} \\
&= 0 - \left((-3)^4 + \frac{8}{3}(-3)^3 - 6(-3)^2 \right) \\
&= -(81 - 72 - 54) \\
&= 45
\end{aligned}$$

$$\begin{aligned}
\int_{0}^{1} (4x^3 + 8x^2 - 12x)\,dx &= x^4 + \frac{8}{3}x^3 - 6x^2 \Big|_{0}^{1} \\
&= \left((1)^4 + \frac{8}{3}(1)^3 - 6(1)^2 \right) - 0 \\
&= -\frac{7}{3}
\end{aligned}$$

Hence the total area is $45 + \frac{7}{3} = \frac{142}{3}$ square units.