 FINAL EXAM

1.

Evaluate $\lim_{x\to-\infty}\left(1-\frac{3}{x}\right)^{2x}$ **8 Marks**

$\lim_{x\to-\infty}\left(1-\frac{3}{x}\right)^{2x}$ is of the form 1^∞

let $L = \lim_{x\to-\infty}\left(1-\frac{3}{x}\right)^{2x}$ then

$$\ln L = \lim_{x\to-\infty} \ln\left(1-\frac{3}{x}\right)^{2x}$$

$$= \lim_{x\to-\infty} 2x \ln\left(1-\frac{3}{x}\right)$$

$$= \lim_{x\to-\infty} \frac{\ln\left(1-\frac{3}{x}\right)}{\frac{1}{2x}} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$\stackrel{H}{=} \lim_{x\to-\infty} \frac{(3/x^2)/\left(1-\frac{3}{x}\right)}{-1/2x^2}$$

$$= \lim_{x\to-\infty} \frac{-6}{1-\frac{3}{x}} = -6$$

Thus, $L = e^{-6}$.

2.

Show that the improper integral $\int_0^2 \frac{dx}{(1-x)^2}$ converges and find its value, or show that it diverges. **8 Marks**

$$\int_0^2 \frac{dx}{(1-x)^2} = \int_0^1 \frac{dx}{(1-x)^2} + \int_1^2 \frac{dx}{(1-x)^2}$$

$$= \lim_{N\to 1^-} \int_0^N \frac{dx}{(1-x)^2} + \lim_{N\to 1^+} \int_N^2 \frac{dx}{(1-x)^2}$$

$$\lim_{N\to 1^-} \int_0^N \frac{dx}{(1-x)^2} = \lim_{N\to 1^-} \left[\frac{-1}{(1-x)}\right]_0^N$$

$$= \lim_{N\to 1^-} \left(\frac{-1}{1-N} + 1\right) = -\infty$$

The integral diverges because the first part diverges.

(b) $\int_{1/3}^{e} 3(\ln 3x)^2\,dx$ **10 Marks**

let $t = 3x$, $dt = 3dx$ and

$$\int_{1/3}^{e} 3(\ln 3x)^2\,dx = \int_{1}^{3e} (\ln t)^2\,dt$$

[Using $u = (\ln t)^2$, $dv = dt$, $du = 2\frac{\ln t}{t}dt$, $v = t$]

Then,

$$\int_1^{3e} (\ln t)^2\,dt = \left[t(\ln t)^2\right]_1^{3e} - 2\int_1^{3e} \ln t\,dt$$

[Using $u = \ln t$, $dv = dt$, $du = \frac{1}{t}dt$, $v = t$]

Therefore,

$$\begin{aligned}\int_1^{3e} (\ln t)^2\,dt &= 3e(\ln 3e)^2 - 2\left(\left[t\ln t\right]_1^{3e} - \int_1^{3e} dt\right)\\ &= 3e(\ln 3 + 1)^2 - 2\left(3e(\ln 3e) - \left[t\right]_1^{3e}\right)\\ &= 3e\left((\ln 3)^2 + 2\ln 3 + 1\right)\\ &\quad - 2\left(3e(\ln 3 + 1) - 3e + 1\right)\\ &= 3e(\ln 3)^2 + 3e - 2\end{aligned}$$

(c) $\int \cos^3 x \sin^4 x \, dx$ 8 Marks

$$\int \cos^3 x \sin^4 x \, dx = \int \cos^2 x \sin^4 x \cos x \, dx$$
$$= \int (1 - \sin^2 x) \sin^4 x \cos x \, dx$$

let $u = \sin x$, then $du = \cos x \, dx$, and

$$\int \cos^3 x \sin^4 x \, dx = \int (1 - u^2) u^4 \, du$$
$$= \int (u^4 - u^6) \, du$$
$$= \frac{1}{5} u^5 - \frac{1}{7} u^7 + C$$
$$= \frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + C \, .$$

(d) $\displaystyle\int \frac{x^2}{(1-9x^2)^{3/2}}\,dx.$ **12 Marks**

let $3x = \sin\theta$ or $x = \frac{1}{3}\sin\theta$, $dx = \frac{1}{3}\cos\theta\,d\theta$.

$$\int \frac{x^2}{(1-9x^2)^{3/2}}dx = \int \frac{(\frac{1}{3}\sin\theta)^2(\frac{1}{3}\cos\theta)}{(1-\sin^2\theta)^{3/2}}d\theta$$

$$= \frac{1}{27}\int \frac{\sin^2\theta\cos\theta}{(\cos^2\theta)^{3/2}}d\theta$$

$$= \frac{1}{27}\int \frac{\sin^2\theta\cos\theta}{\cos^3\theta}d\theta$$

$$= \frac{1}{27}\int \tan^2\theta\,d\theta$$

$$= \frac{1}{27}\int (\sec^2\theta - 1)\,d\theta$$

$$= \frac{1}{27}(\tan\theta - \theta) + C$$

$$= \frac{x}{9\sqrt{1-9x^2}} - \frac{1}{27}\sin^{-1}(3x) + C$$

(e) $\int \frac{2x^3+10x}{(x^2+1)^2}\,dx.$ **12 Marks**

$$\frac{2x^3+10x}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} \Rightarrow$$

$$2x^3+10x = (Ax+B)(x^2+1) + Cx + D$$

Equating x^3 terms: $2 = A$

Equating x^2 terms: $0 = B$

Equating x^1 terms: $10 = A + C \Rightarrow C = 8$

Equating x^0 terms: $0 = B + D \Rightarrow D = 0$. Thus,

$$\int \frac{2x^3+10x}{(x^2+1)^2}\,dx = \int \left[\frac{2x}{x^2+1} + \frac{8x}{(x^2+1)^2}\right] dx$$

$$= \ln(x^2+1) - \frac{4}{x^2+1} + K$$

4.
Find the length of the curve defined by $(y+1)^2 = 4x^3$ between the points $(0,-1)$ and $(1,1)$.
7 Marks

$y = f(x) = -1 + 2x^{3/2} \Rightarrow f'(x) = 3x^{1/2}$

$1 + [f'(x)]^2 = 1 + 9x$. Then,

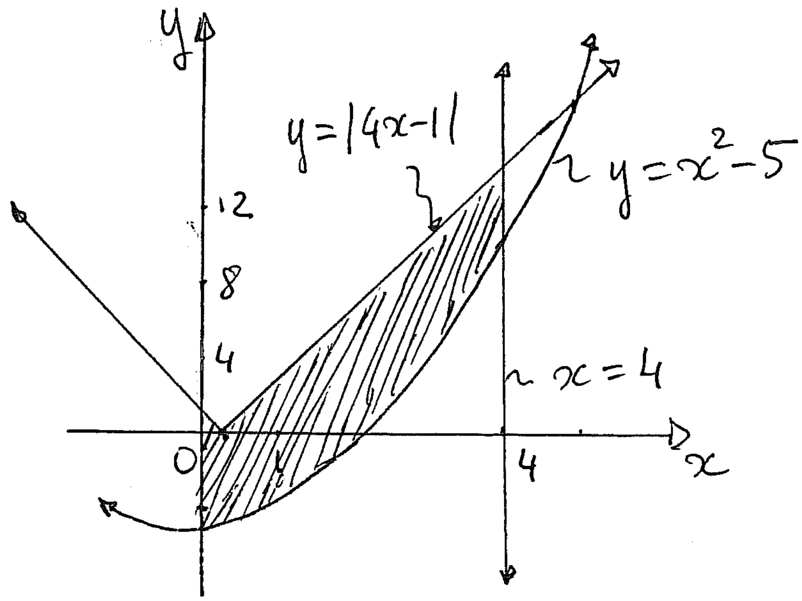
$$L = \int_0^1 (1+9x)^{1/2}\,dx$$

$$= \frac{2}{27}\left[(1+9x)^{3/2}\right]_0^1 = \frac{2}{27}\left(10\sqrt{10} - 1\right).$$

5.
For each of the following region, give a rough sketch of the region described, and set up a definite integral (or combination of definite integrals) to find the area of the region. **DO NOT SOLVE THE DEFINITE INTEGRALS.**

(a) The region between the curves $y = |4x - 1|$ and $y = x^2 - 5$, $x = 0$, $x = 4$.

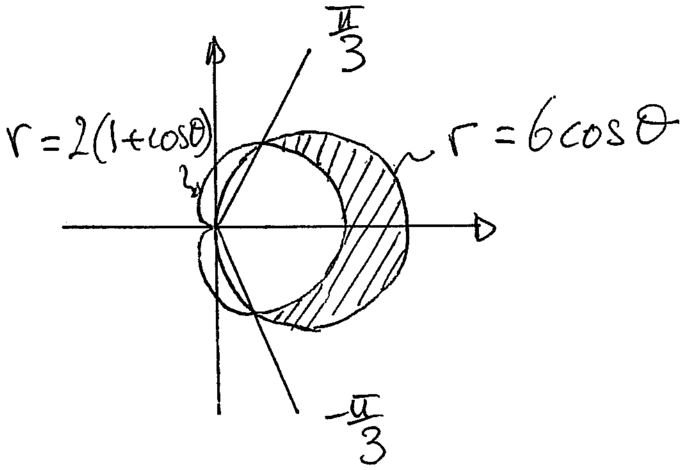
6 Marks



The curves $y=|4x-1|$ and $y=x^2-5$ do not intersect on $[0,4]$, but the absolute value function causes the equation of the line to change at $x=\frac{1}{4}$.

$$A = \int_0^{1/4} \left(-4x+1-(x^2-5)\right)dx + \int_{1/4}^{4} \left(4x-1-(x^2-5)\right)dx$$

(b) The region (described in polar coordinates) inside the circle $r = 6\cos\theta$ and outside the cardioid $r = 2(1+\cos\theta)$. **6 Marks**



For intersections:

$6\cos\theta = 2+2\cos\theta$

$4\cos\theta = 2$

$\cos\theta = \frac{1}{2}$

$\theta = \pm\frac{\pi}{3}$

The shaded area is given by

$$2\times\frac{1}{2}\int_0^{\pi/3}\left[36\cos^2\theta - 4(1+\cos\theta)^2\right]d\theta.$$

6.
Set up **BUT DO NOT SOLVE** the integrals which give the volume of solid obtained by rotating the region bounded by $y = 4 - x^2$, the x-axis and the y-axis:

(a) about the x-axis. **5 Marks**

(b) about the y-axis. **5 Marks**

(c) about the line $x = -2$. **5 Marks**

(a) shells : $2\pi \int_0^4 y\sqrt{4-y}\,dy$

(b) disks : $\pi \int_0^4 (4-y)\,dy$

(c) washers : $\pi \int_0^4 \left[(x+2)^2 - 2^2\right] dy$

$$= \pi \int_0^4 \left[4 - y + 4\sqrt{4-y}\right] dy.$$