1.

Evaluate $\lim_{x\to 0^+} (1+3x)^{\csc x}$ 8 Marks

$\lim_{x\to 0^+} (1+3x)^{\csc x}$ is of the form 1^{∞}.

Let $L = \lim_{x\to 0^+} (1+3x)^{\csc x}$ then

$$\ln L = \lim_{x\to 0^+} \ln (1+3x)^{\csc x}$$

$$= \lim_{x\to 0^+} \csc x \ln(1+3x)$$

$$= \lim_{x\to 0^+} \frac{\ln(1+3x)}{\sin x} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$\overset{H}{=} \lim_{x\to 0^+} \frac{3/(1+3x)}{\cos x} = \frac{3}{1} = 3$$

Thus, $L = e^3$.

2.

Determine whether the improper integral $\int_{-\infty}^{0} \frac{dx}{(x-8)^{2/3}}$ converges or diverges. If it converges, find its value.

8 Marks

$$\int_{-\infty}^{0} \frac{dx}{(x-8)^{2/3}} = \lim_{t \to -\infty} \int_t^0 (x-8)^{-2/3}\, dx$$

$$= \lim_{t \to -\infty} \left[3(x-8)^{1/3} \right]_t^0$$

$$= 3 \lim_{t \to -\infty} \left[-2 - (t-8)^{1/3} \right]$$

$$= \infty$$

It diverges.

3.

Solve the following integrals:

(a) $\int_0^4 x^3(x^2+1)^{-1/2}\,dx$ **8 Marks**

Let $u = x^2+1$, $x^2 = u-1$ and $du = 2x\,dx$.

$$\int_0^4 x^3(x^2+1)^{-1/2}\,dx = \frac{1}{2}\int_1^{17}(u-1)u^{-1/2}\,du$$

$$= \frac{1}{2}\left[\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right)\right]_1^{17}$$

$$= \frac{17\sqrt{17}-1}{3} - \left(\sqrt{17}-1\right)$$

$$= \frac{14\sqrt{17}}{3} + \frac{2}{3}.$$

(b) $\int x^2 e^{3x}\,dx$ **10 Marks**

$$\int x^2 e^{3x}\,dx = \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\int x e^{3x}\,dx$$

[Using $u = x^2$, $du = 2x\,dx$, $dv = e^{3x}\,dx$, $v = \frac{1}{3}e^{3x}$]

Then,

$$\int x^2 e^{3x}\,dx = \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\left[\frac{1}{3}x e^{3x} - \frac{1}{3}\int e^{3x}\,dx\right]$$

[using $u = x$, $du = dx$, $dv = e^{3x}\,dx$, $v = \frac{1}{3}e^{3x}$]

Therefore,

$$\int x^2 e^{3x}\,dx = \frac{1}{3}x^2 e^{3x} - \frac{2}{9}x e^{3x} + \frac{2}{27}e^{3x} + C$$

$$= \frac{1}{27}e^{3x}\left(9x^2 - 6x + 2\right) + C.$$

(c) $\int \sin^5 x \cos^3 x \, dx$ 8 Marks

$$\int \sin^5 x \cos^3 x \, dx = \int \sin^5 x \cos^2 x \cos x \, dx$$

$$= \int \sin^5 x (1 - \sin^2 x) \cos x \, dx.$$

let $u = \sin x$, then $du = \cos x \, dx$, and

$$\int \sin^5 x \cos^3 x \, dx = \int u^5 (1 - u^2) \, du$$

$$= \int (u^5 - u^7) \, du$$

$$= \frac{1}{6} u^6 - \frac{1}{8} u^8 + C$$

$$= \frac{1}{6} \sin^6 x - \frac{1}{8} \sin^8 x + C.$$

(d) $\int \frac{1}{x^3\sqrt{x^2-25}}\,dx$. **12 Marks**

Let $x = 5\sec\theta$, $dx = 5\sec\theta\tan\theta\,d\theta$. Then,

$$\int \frac{1}{x^3\sqrt{x^2-25}}\,dx = \int \frac{5\sec\theta\tan\theta}{(125\sec^3\theta)(5\tan\theta)}\,d\theta$$

$$= \frac{1}{125}\int \cos^2\theta\,d\theta$$

$$= \frac{1}{125}\int \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{250}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{250}\left(\theta + \sin\theta\cos\theta\right) + C$$

$$= \frac{1}{250}\left[\sec^{-1}\left(\frac{x}{5}\right) + \frac{\sqrt{x^2-25}}{x}\cdot\frac{5}{x}\right] + C$$

$$= \frac{1}{250}\left[\sec^{-1}\left(\frac{x}{5}\right) + \frac{5\sqrt{x^2-25}}{x^2}\right] + C$$

(e) $\displaystyle\int \frac{2x^2-25x-33}{(x+1)^2(x-5)}\,dx.$ **12 Marks**

$$\frac{2x^2-25x-33}{(x+1)^2(x-5)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x-5)} \Rightarrow$$

$$2x^2-25x-33 = A(x+1)(x-5) + B(x-5) + C(x+1)$$

Let $x=-1$, then $B=1$.

Let $x=5$, then $C=-3$.

Equating x^2 terms: $2 = A + C \Rightarrow A = 5$. Thus,

$$\int \frac{2x^2-25x-33}{(x+1)^2(x-5)}\,dx = \int \left[\frac{5}{x+1} + \frac{1}{(x+1)^2} + \frac{-3}{x-5}\right] dx$$

$$= 5\ln|x+1| - \frac{1}{x+1} - 3\ln|x-5| + K$$

4.
Find the length of the curve defined by $(y+1)^2 = (x-4)^3$ between the points $(5,0)$ and $(8,7)$.
7 Marks

$(y+1)^2 = (x-4)^3$ and $y+1 \geq 0$ from A to B $\Rightarrow$

$y = f(x) = -1 + (x-4)^{3/2}$

$1 + [f'(x)]^2 = 1 + \left[\frac{3}{2}(x-4)^{1/2}\right]^2 = \frac{9}{4}x - 8$

Then,

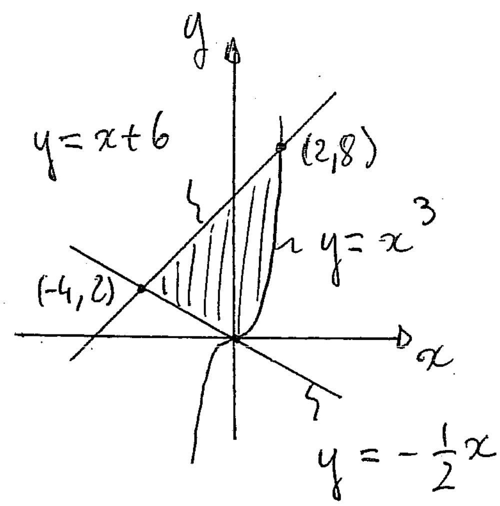
$$L = \int_5^8 \left(\frac{9}{4}x - 8\right)^{1/2} dx = \frac{8}{27}\left[\left(\frac{9}{4}x - 8\right)^{3/2}\right]_5^8$$

$$= \frac{8}{27}\left[10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right]$$

5.
For each of the following region, give a rough sketch of the region described, and set up a definite integral (or combination of definite integrals) to find the area of the region. DO NOT SOLVE THE DEFINITE INTEGRALS.

(a) The region between the curves $y = x + 6$, $y = x^3$, and $2y + x = 0$.

6 Marks

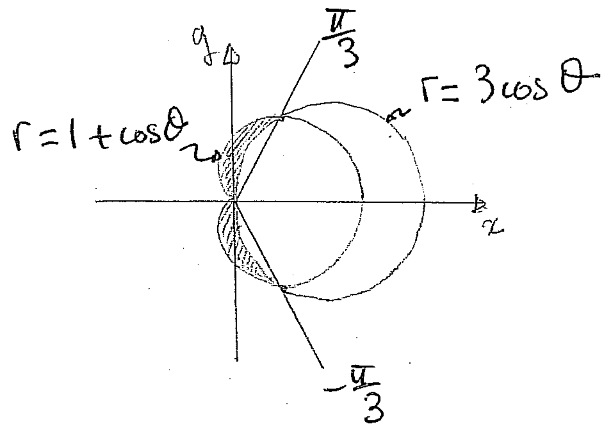


The shaded area is given by

$$\int_{-4}^{0} \left[(x+6) - \left(-\frac{1}{2}x\right)\right] dx + \int_{0}^{2} \left[(x+6) - x^3\right] dx .$$

$$\int_{0}^{2} \left[\sqrt[3]{y} - (-2y)\right] dy + \int_{2}^{8} \left[\sqrt[3]{y} - (y-6)\right] dy$$

(b) The region (described in polar coordinates) inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 3\cos\theta$. **6 Marks**



For intersections:

$1 + \cos\theta = 3\cos\theta$.

Thus, $2\cos\theta = 1$

and $\theta = \pm\frac{\pi}{3}$.

The shaded area is given by

$$2 \times \frac{1}{2}\left[\int_{\pi/3}^{\pi} (1+\cos\theta)^2\, d\theta - 9\int_{\pi/3}^{\pi/2} \cos^2\theta\, d\theta\right]$$

6.
Set up **BUT DO NOT SOLVE** the integrals which give the volume of solid obtained by rotating the region bounded by $y = 4 - x$, $x = 0$, and $y = 0$ with $0 \leq x \leq 4$:

(a) about the x-axis. **5 Marks**

(b) about the y-axis. **5 Marks**

(c) about the line $y = -1$. **5 Marks**

(a) disks: $\pi \int_0^4 (4-x)^2 \, dx$

(b) shells: $2\pi \int_0^4 x(4-x) \, dx$

(c) washers: $\pi \int_0^4 \left[(y+1)^2 - 1^2\right] dx$

$$= \pi \int_0^4 \left[24 - 10x + x^2\right] dx.$$