MATH 1700 D01 Fall 2017 Assignment 1

SHOW ALL WORK to get full marks. Leave answers as exact answers. For example, leave it as fractions such as 1/7 as opposed to decimals such as 0.142857. Word problems should have sentence answers with units. Fractions should be lowest terms.

Calculators are not permitted. Assignments using a calculator will not be graded.

All assignments must be handed in on UMLearn as **one PDF file**. Late assignments will not be accepted. Failure to follow the instructions will result in a mark of 0.

Techniques from this course must be used to solve the questions, not more advanced techniques.

The assignment covers sections 4.4,1.5,3.5,10.1–10.3, 5.1–5.2 in the textbook.

Assignments are to be done independently.

1. Evaluate the following limits using L'Hopital's Rule. Make sure to justify that you can use L'Hopital's Rule every time.

[3] (a) $\lim\limits_{x\to 1} \dfrac{\tan(x-1)}{x^4-1}$

Solution:
Plugging in $x = 1$ gives a 0/0 form so L'Hopital's Rule applies. Therefore

$$\lim_{x\to 1} \frac{\tan(x-1)}{x^4-1} = \lim_{x\to 1} \frac{\sec^2(x-1)}{4x^3} = \frac{\sec^2(0)}{4(1)^3} = \frac{1}{4}.$$

[3] (b) $\lim\limits_{x\to 0} \dfrac{\cos(x^2)-1}{x^4}$

Solution:
Plugging in $x \to 0$ gives a 0/0 form so L'Hopital's Rule does apply.

$$\lim_{x\to 0} \frac{-2x\sin(x^2)}{4x^3} = \lim_{x\to 0} \frac{-\sin(x^2)}{2x^2}$$

which is still a 0/0 form so L'Hopital's Rule does apply.

$$\lim_{x\to 0} \frac{-2x\cos(x^2)}{4x} = \lim_{x\to 0} \frac{-1\cos(0)}{2} = -\frac{1}{2}.$$

[4] (c) $\lim\limits_{x\to -\infty} (x\sec(2/x) - x)$

Solution:
This is an $\infty - \infty$ indeterminant form for which L'Hopital's Rule does not apply.

However we can turn it into a fraction by noting

$$\lim_{x\to-\infty} x\sec(2/x) - x = \lim_{x\to-\infty} x(\sec(2/x)-1) = \lim_{x\to-\infty} \frac{\sec(2/x)-1}{1/x}$$

which is a 0/0 form and therefore L'Hopital's Rule does apply.

$$\begin{aligned}\lim_{x\to-\infty} \frac{\sec(2/x)-1}{1/x} &= \lim_{x\to-\infty} \frac{\sec(2/x)\tan(2/x)(-2x^{-2})}{-x^{-2}} \\ &= \lim_{x\to-\infty} 2\sec(2/x)\tan(2/x) = 2\sec 0\tan 0 = 0.\end{aligned}$$

[5] (d) $\lim_{x\to0^+}(\tan x)^{\sqrt{x}}$

Solution:
This is a 0^0 indeterminant form. So let $L = \lim_{x\to0^+}(\tan x)^{x^{1/2}}$, and then

$$\ln L = \lim_{x\to0^+} x^{1/2}\ln(\tan x) = \lim_{x\to0^+}\frac{\ln(\tan x)}{x^{-1/2}}$$

which is an $-\infty/\infty$ form so L'Hopital's rule applies.

$$\ln L = \lim_{x\to0^+}\frac{\ln(\tan x)}{x^{-1/2}} = \lim_{x\to0^+}\frac{1/(\sin x\cos x)}{-\frac{1}{2}x^{-3/2}} = \lim_{x\to0^+}\left(-\frac{2x^{3/2}}{\sin x\cos x}\right)$$

which is a 0/0 form so L'Hopital's rule applies.

$$\ln L = \lim_{x\to0^+}\left(-\frac{2x^{3/2}}{\sin x\cos x}\right) = \lim_{x\to0^+}\left(-\frac{3x^{1/2}}{\cos^2 x-\sin^2 x}\right) = 0.$$

Hence

$$L = e^{\ln L} = e^0 = 1.$$

[5] (e) $\lim_{x\to0}(e^x+2x)^{3/x}$

Solution:
This is a 0^0 indeterminant form. So let $L = \lim_{x\to0}(e^x+2x)^{3/x}$, and then

$$\ln L = \lim_{x\to0}\frac{3}{x}\ln(e^x+2x) = \lim_{x\to0^+}\frac{3\ln(e^x+2x)}{x}$$

which is a 0/0 form so L'Hopital's rule applies.

$$\ln L = \lim_{x\to 0^+} \frac{3\ln(e^x+2x)}{x} = \lim_{x\to 0^+} \frac{\frac{3(e^x+2)}{e^x+2x}}{1} = \lim_{x\to 0^+} \frac{3(e^x+2)}{e^x+2x} = 9.$$

Hence

$$L = e^{\ln L} = e^9.$$

[6] 2. Simplify the expression $\csc(\tan^{-1} x)$. Make sure you are showing full work and explain your answer.

Solution:

Suppose $\theta = \tan^{-1} x$. Then $x = \csc\theta$ where θ is in the interval $(-\pi/2, \pi/2)$.

Solution 1: Using the identity

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\tan\theta\cos\theta} = \frac{\sec\theta}{\tan\theta}$$

Then $\sec^2\theta = 1 + \tan^2\theta = 1 + x^2 \Rightarrow \sec\theta = \pm\sqrt{1+x^2}$.

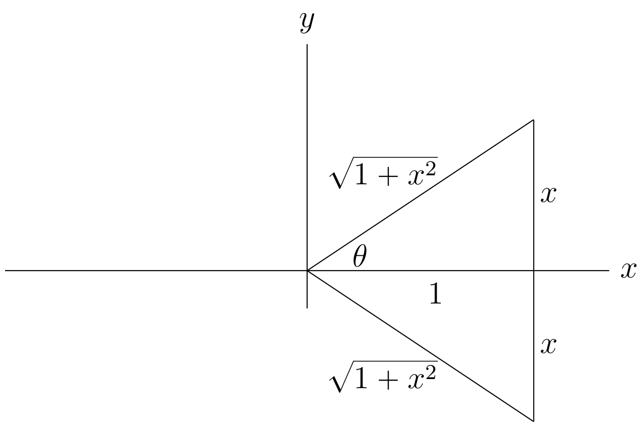
However, since θ is in the range of $(-\pi/2\,\pi/2)$ and $\sec\theta = \dfrac{1}{\cos\theta}$ is positive over that interval, we have that $\sec\theta = \sqrt{1+x^2}$.

Therefore

$$\csc(\tan^{-1} x) = \csc\theta = \frac{\sqrt{1+x^2}}{x}$$

Solution 2:

$\tan\theta = \dfrac{x}{1}$ in quadrants I and IV. A sketch of both possibilities are given below.



where we get $\sqrt{1+x^2}$ from the Pythagorean Theorem. Now we are looking for $\csc\theta = \dfrac{1}{\sin\theta} = \dfrac{1}{\frac{x}{\sqrt{1+x^2}}} = \dfrac{\sqrt{1+x^2}}{x}$.

Hence

$$\csc(\tan^{-1} x) = \csc\theta = \frac{\sqrt{1+x^2}}{x}.$$

3. For the parametric curve $x = t^2 + t, y = t^3 + t^2$

[2] (a) Compute dy/dx in terms of t.

Solution:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 + 2t}{2t+1}.$$

[5] (b) Determine the equation of the tangent line to the curve at the point $(2, -4)$.

Solution: The slope is $\left.\dfrac{dy}{dx}\right|_{t=t_0}$ where t_0 is the value of t which gives the point $(2, -4)$. If

$$x = 2 \Rightarrow t^2 + t = 2,\ 0 = t^2 + t - 2 = (t+2)(t-1) \Rightarrow t = -2, 1.$$

Testing in the x equation we see that $t = -2$ makes $y = -4$ which is true and using $t = 1$ makes $y = 2$ which is false. Therefore $t = -2$. Hence the slope is

$$m = \frac{3(-2)^2 + 2(-2)}{2(-2)+1} = \frac{8}{-3} = -\frac{8}{3}.$$

Therefore the tangent line is $y + 4 = -\frac{8}{3}(x - 2)$.

[4] (c) Find the point(s) on the curve where the tangent line to the curve at that point is horizontal.

Solution: The tangent to the curve is horizontal if $dy/dt = 0$ and $dx/dt \neq 0$. Since $dy/dt = 3t^2 + 2t = 0$ when $t = 0, -2/3$ neither of which make $dx/dt = 0$. Thus we have that the curve is horizontal when $t = 0, -2/3$. To find the points, we plug in the value into x and y to get (0, 0) and (−2/9, 4/27).

[5] (d) Find the point(s) on the curve where the tangent line to the curve at that point is vertical.

Solution: The tangent to the curve is vertical if $dx/dt = 0$ and $dy/dt \neq 0$. Since $dx/dt = 2t + 1 = 0 \Rightarrow t = -1/2$ which does not make $dy/dt = 0$. Thus we have that the curve is vertical when $t = \pm - 1/2$. To find the points, we plug in the values into x and y to get (−1/4, 1/8).

4. For the curve $x = t^2 + t, y = t^3 + t^2$

[7] (a) Show the curve crosses itself at the point (0, 0). (Hint: Find two different values of t which give the same point and show that the tangent lines at those values are different)

Solution: The curve has $x = 0$ when $t = -1, 0$. At both these points, $y = 0$.

Technically this doesn't prove that the curve crosses itself, as the function could just be periodic. For example the parametric curve representing a circle $x = \cos t$, $y = \sin t$ passes through $(1, 0)$ over and over again, however we wouldn't say the circle crosses itself.

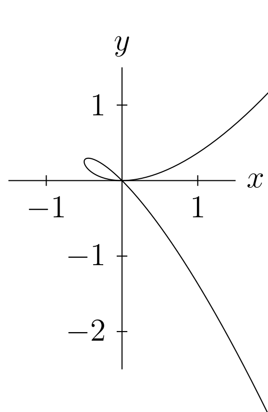
So to show the curve crosses itself, we need to show that the tangent lines at $t = \pm 1$ are different.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 + 2t}{2t + 1}.$$

When $t = 0$ this is 0 and when $t = -1$ this is −1 Hence the curve crosses itself at (0, 0).

[4] (b) Sketch the curve using information from question 3 and 4a.

Solution:



[6] 5. Find $\dfrac{dy}{dx}$ for the polar curve $r = 3+3\sin\theta$ and use it to find the points where the tangent line is vertical,horizontal or neither when $\theta = \dfrac{3\pi}{2}$.

Solution:

We know that $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$. Hence using

$$x = r\cos\theta = (3+3\sin\theta)\cos\theta = 3\cos\theta + 3\cos\theta\sin\theta$$

and

$$y = r\sin\theta = (3+3\sin\theta)\sin\theta = 3\sin\theta + 3\sin^2\theta$$

yields

$$\frac{dy}{dx} = \frac{3\cos\theta + 6\sin\theta\cos\theta}{-3\sin\theta + 3\cos^2\theta - 3\sin^2\theta}$$

When $\theta = 3\pi/2$ we have that both the numerator and the denominator is 0. Hence we must take limits (and L'Hopital's rule on the 0/0 form) to see that

$$\lim_{\theta\to 3\pi/2} \frac{-3\sin\theta - 6\cos^2\theta - +6\sin^2\theta}{-3\sin\theta - 12\sin\theta\cos\theta}$$

which is a non-zero over zero form and thus goes towards $\pm\infty$. Hence the tangent line when $\theta = 3\pi/2$ is vertical.

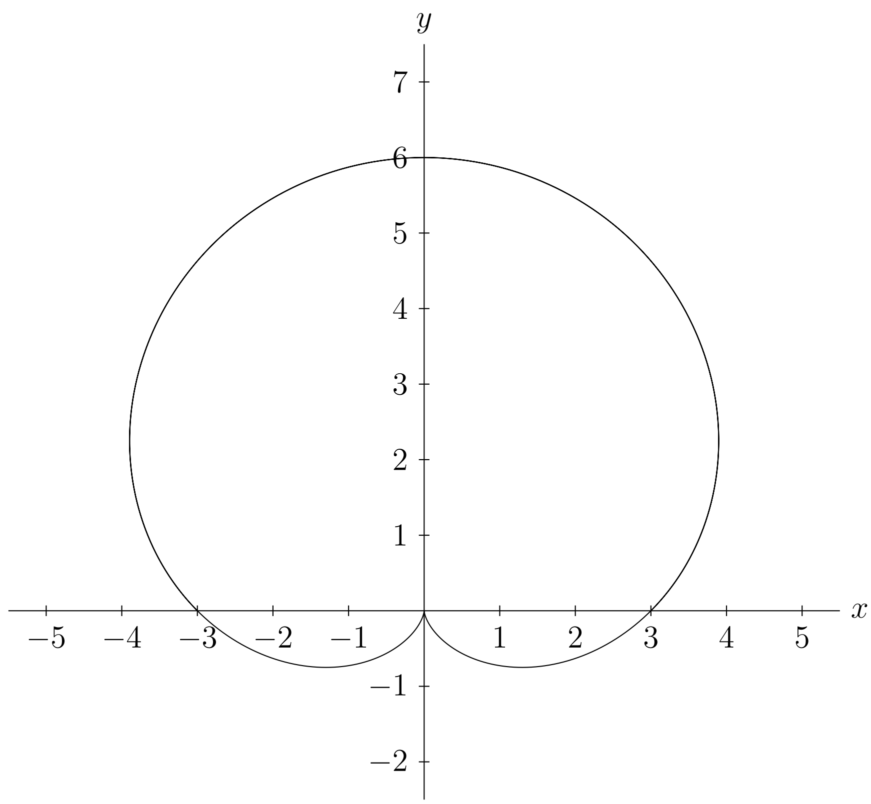
6. Sketch the polar curves

[3] (a) $r = 3 + 3\sin\theta$

Solution:

- As θ goes from 0 to $\pi/2$, r goes from 3 to 6
- As θ goes from $\pi/2$ to π, r goes from 6 to 3
- As θ goes from π to $3\pi/2$, r goes from 3 to 0
- As θ goes from $3\pi/2$ to 2π, r goes from 0 to 3.

Hence the graph is



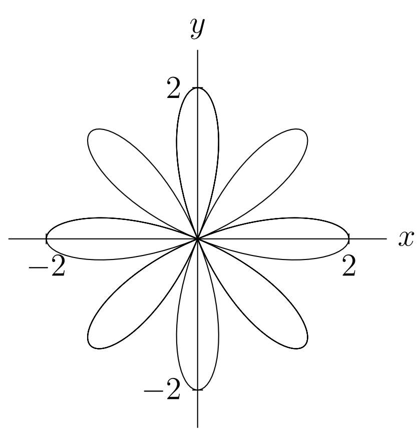
[3] (b) $r = 2\cos 4\theta$.

Solution:

- As θ goes from 0 to $\pi/8$, r goes from 2 to 0
- As θ goes from $\pi/8$ to $\pi/4$, r goes from 0 to -2
- As θ goes from $\pi/4$ to $3\pi/8$, r goes from -2 to 0

- As θ goes from $3\pi/8$ to $\pi/2$, r goes from 0 to 2
- As θ goes from $\pi/2$ to $5\pi/8$, r goes from 2 to 0
- As θ goes from $5\pi/8$ to $3\pi/4$, r goes from 0 to -2
- As θ goes from $3\pi/4$ to $7\pi/8$, r goes from -2 to 0
- As θ goes from $7\pi/8$ to π, r goes from 0 to 2
- As θ goes from π to $9\pi/8$, r goes from 2 to 0
- As θ goes from $9\pi/8$ to $5\pi/4$, r goes from 0 to -2
- As θ goes from $5\pi/4$ to $11\pi/8$, r goes from -2 to 0
- As θ goes from $11\pi/8$ to $3\pi/2$, r goes from 0 to 2
- As θ goes from $3\pi/2$ to $13\pi/8$, r goes from 2 to 0
- As θ goes from $13\pi/8$ to $7\pi/4$, r goes from 0 to -2
- As θ goes from $7\pi/4$ to $15\pi/8$, r goes from -2 to 0
- As θ goes from $15\pi/8$ to 2π, r goes from 0 to 2

Hence the graph is



[10] 7. Use the Riemann sum compute the area under the curve of $y = x^3 + 3x - 3$ from $x = 1$ to $x = 4$. (Note: $f(x)$ is positive on $[1, 4]$)

Solution: The Riemann sum definition is

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where x_i^* is any point in the subinterval from x_{i-1} to x_i. Choosing the right hand endpoints yields $x_i^* = x_i$.

$$\Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}, f(x) = x^3 + 3x - 3 \text{ and } x_i^* = x_i = a + i\Delta x = 1 + \frac{3i}{n}.$$

Hence the area is

$$\begin{aligned}
A &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1 + \frac{3i}{n}\right)\frac{3}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(1 + \frac{3i}{n}\right)^3 + 3\left(1 + \frac{3i}{n}\right) - 3\right)\frac{3}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{9i}{n} + \frac{27i^2}{n^2} + \frac{27i^3}{n^3} + 3 + \frac{9i}{n} - 3\right)\frac{3}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{18i}{n} + \frac{27i^2}{n^2} + \frac{27i^3}{n^3}\right)\frac{3}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3}{n} + \frac{54}{n^2} + \frac{81i^2}{n^3} + \frac{81i^3}{n^4}\right)
\end{aligned}$$

Using that

$$\sum_{i=1}^{n} 1 = n, \sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

we get the area

$$\begin{aligned}
A &= \lim_{n\to\infty} \left(\frac{3n}{n} + \frac{54n(n+1)}{2n^2} + \frac{81n(n+1)(2n+1)}{6n^3} + \frac{81n^2(n+1)^2}{4n^4}\right) \\
&= \lim_{n\to\infty} \left(3 + \frac{27(n+1)}{n} + \frac{27(2n^3+3n^2+n)}{2n^3} + \frac{81(n^4+2n^3+n^2)}{4n^4}\right) \\
&= \lim_{n\to\infty} \left(3 + 27 + \frac{27}{n} + 27 + \frac{81}{2n} + \frac{27}{2n^2} + \frac{81}{4} + \frac{81}{2n} + \frac{81}{4n^2}\right) \\
&= 3 + 27 + 0 + 27 + 0 + 0 + \frac{81}{4} + 0 + 0 \\
&= \frac{309}{4}.
\end{aligned}$$

This assignment is out of 75 points.