MATH 1700 Assignment 1 (Sol-ns)

#1 a) $\lim_{x\to 0} \frac{\sqrt{5+x}-\sqrt{5-x}}{x} = \left\{ \frac{\sqrt{5}-\sqrt{5}}{0} = \frac{0}{0} \quad \therefore \text{L'HR} \right\} =$

$= \lim_{x\to 0} \frac{(\sqrt{5+x}-\sqrt{5-x})'}{x'} =$

$= \lim_{x\to 0} \frac{\frac{1}{2}(5+x)^{-\frac{1}{2}} - (-1)\frac{1}{2}(5-x)^{-\frac{1}{2}}}{1} =$

$= \lim_{x\to 0} \left(\frac{1}{2\sqrt{5+x}} + \frac{1}{2\sqrt{5-x}} \right) =$

$= \frac{1}{2\sqrt{5}} + \frac{1}{2\sqrt{5}} = \frac{2}{2\sqrt{5}} = \boxed{\frac{1}{\sqrt{5}}}$

Answer

b) $\lim_{x\to \frac{\pi}{2}} \frac{\tan x}{x} = \left\{ \frac{\tan\frac{\pi}{2}}{\frac{\pi}{2}} = \frac{\infty}{\pi/2} \right.$ $\therefore$ l'Hospital's Rule CANNOT Be used $\Big\}$

Since $\lim_{x\to\pi/2} \tan x$ does not exist (because $\lim_{x \to \left(\frac{\pi}{2}\right)^-} \tan x = +\infty$ and

$\lim_{x \to \left(\frac{\pi}{2}\right)^+} \tan x = -\infty$), we

conclude that $\boxed{\lim_{x \to \frac{\pi}{2}} \frac{\tan x}{x} \quad \text{DNE}}$

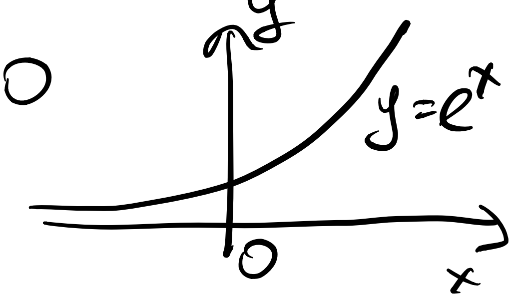
c) $\lim_{x \to -\infty} xe^x = \left\{ -\infty \cdot e^{-\infty} = -\infty \cdot 0 \right.$ ∴ NOT the right form $\left. \right\}$

$= \lim_{x \to -\infty} \frac{x}{e^{-x}} = \left\{ \frac{-\infty}{e^{\infty}} = \frac{\infty}{\infty} \right.$ ∴ L'HR $\left. \right\} =$

$= \lim_{x \to -\infty} \frac{x'}{(e^{-x})'} = \lim_{x \to -\infty} \frac{1}{-e^{-x}} =$

$= \lim_{x \to -\infty} (-e^x) = \boxed{0}$, Answer

since $\lim_{x \to -\infty} e^x = 0$



d) $\lim_{y\to\infty} y \sin\frac{1}{y} = \left\{ \begin{array}{l} \infty \cdot \sin 0 = \infty \cdot 0 \\ \therefore \text{NOT the right form} \end{array} \right\}$

$= \lim_{y\to\infty} \dfrac{\sin\frac{1}{y}}{\frac{1}{y}} = \left\{ \dfrac{\sin 0}{0} = \dfrac{0}{0} \quad \therefore \text{L'HR} \right\} =$

$= \lim_{y\to\infty} \dfrac{\left(\sin\frac{1}{y}\right)'}{\left(1/y\right)'} = \lim_{y\to\infty} \dfrac{-\frac{1}{y^2}\cos\frac{1}{y}}{-1/y^2} =$

$= \lim_{y\to\infty} \cos\frac{1}{y} = \cos 0 \boxed{= 1}.$

Answer

e) $\lim_{w\to 0} (1+w)^{\cot w} = \left\{ \begin{array}{l} 1^{\infty} \\ \text{NOT the right form} \end{array} \right\}$

$= \left\{ \text{Use: } \boxed{a^b = e^{b \ln a}} \leftarrow \text{USEFUL} \right\}$

$= \lim_{w\to 0} e^{\cot w \cdot \ln(1+w)} =$

$$= e^{\lim_{w\to 0} \cot w \cdot \ln(1+w)} = e^L,$$

and we now need to find L.

$$L = \lim_{w\to 0} \frac{\ln(1+w)}{\tan w} = \left\{ \frac{\ln 1}{\tan 0} = \frac{0}{0} \quad \therefore L'H R \right\} =$$

$$= \lim_{w\to 0} \frac{\frac{1}{1+w}}{\sec^2 w} = 1$$

$$\therefore e^L = e^1 = \boxed{e}$$ Answer

#2. $x = \sqrt{-t^2+2t+3}$

$$y = \frac{1}{-t^2+2t+3}$$

a) $-t^2+2t+3 = x^2$

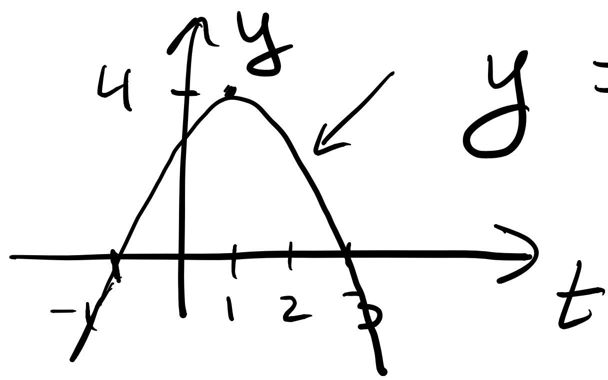
$$\therefore \boxed{y = \frac{1}{x^2}}.$$

b) Since $-t^2+2t+3 \geq 0$, we have

$t^2-2t-3 \leq 0 \iff (t-1)^2-4 \leq 0 \iff$

$\iff (t-1)^2 \leq 4 \iff |t-1| \leq 2 \iff$

$\iff -2 \leq t-1 \leq 2 \iff -1 \leq t \leq 3$



$y = -t^2+2t+3$

$\therefore -1 \leq t \leq 3$ and $-t^2+2t+3$

ranges from 0 to 4

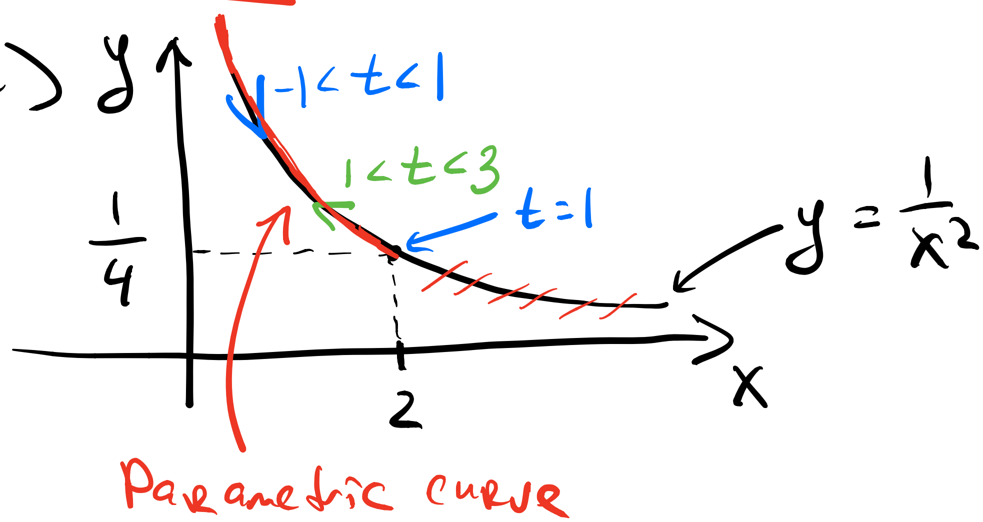
$\therefore$ The range of x as a fn of

t is $[0, 2]$.

Since y is not defined if

$-t^2+2t+3=0$, all values of x for which the eq-n $y=\frac{1}{x^2}$ represents this curve are

$\boxed{(0,2]}$. Answer

c) 

#3. $x=\frac{u}{u-1}$ and $y=\frac{u^2}{u^2-1}$

$$\frac{dy}{dx} = \frac{\frac{dy}{du}}{\frac{dx}{du}} = \frac{\frac{d}{du}\left[\frac{u^2}{u^2-1}\right]}{\frac{d}{du}\left[\frac{u}{u-1}\right]} =$$

$$= \frac{\dfrac{2u(u^2-1) - u^2 \cdot 2u}{(u^2-1)^2}}{\dfrac{1 \cdot (u-1) - u \cdot 1}{(u-1)^2}} =$$

$$= \frac{2u^3 - 2u - 2u^3}{(u^2-1)^2} \cdot \frac{(u-1)^2}{u-1-u} =$$

$$= \frac{-2u}{\cancel{(u-1)^2}(u+1)^2} \cdot \frac{\cancel{(u-1)^2}}{-1} =$$

$$\boxed{= \frac{2u}{(u+1)^2}}$$ Answer

#4. $x = \sqrt{t-1}$, $y = (t^3+1)^{3/2}$

at the pt. $(1, 27)$ (i.e, $t = 2$)

Slope $m = \frac{dy}{dx}$ at $t=2$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{3}{2}(t^3+1)^{1/2} \cdot 3t^2}{\frac{1}{2}(t-1)^{-1/2}} =$$

$$= 3\sqrt{t^3+1} \cdot 3t^2 \cdot \sqrt{t-1}.$$

$$\therefore m = \left.\frac{dy}{dx}\right|_{t=2} = 9 \cdot \sqrt{9} \cdot 2^2 =$$

$$= 9 \cdot 3 \cdot 4 = 108$$

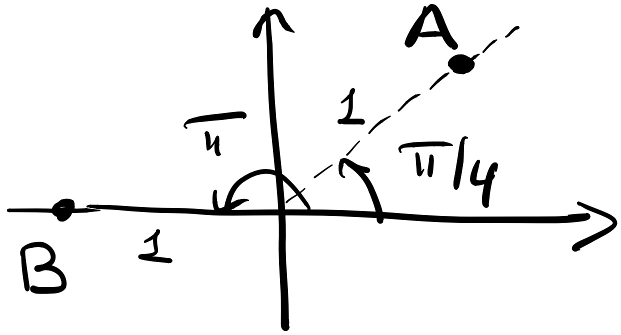
General eq-n of the line through (x_0, y_0) with slope m is

$$y - y_0 = m(x - x_0).$$

$\therefore$ Eq-n of tangent is

$$\boxed{y - 27 = 108(x-1)}$$ Answer

#5. $(r,\theta) = \underbrace{\left(1, \pi/4\right)}_{A}$ and $\underbrace{(1, \pi)}_{B}$.



Find Euclidean coordinates of these points using

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$A: \begin{cases} x = 1\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \\ y = 1\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \end{cases}$$

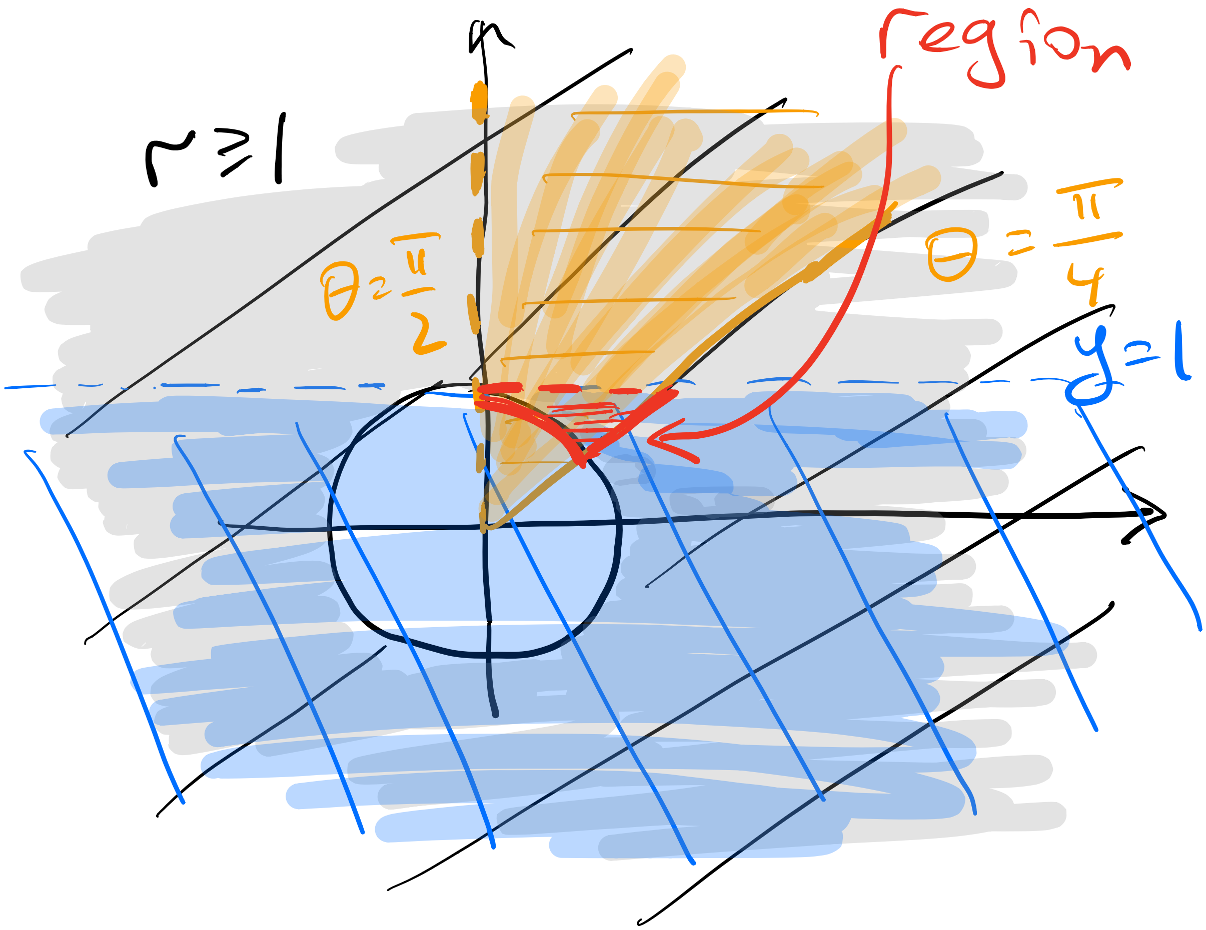
$$B: \begin{cases} x = 1\cos\pi = -1 \\ y = 1\sin\pi = 0 \end{cases}$$

$$|AB| = \sqrt{\left(\frac{1}{\sqrt{2}}+1\right)^2 + \left(\frac{1}{\sqrt{2}}-0\right)^2} = \sqrt{\frac{1}{2}+1+\sqrt{2}+\frac{1}{2}} =$$

$$= \boxed{\sqrt{2+\sqrt{2}}}$$ Answer

#6. $1 \le r$, $r\sin\theta < 1$,

$\frac{\pi}{4} \le \theta < \frac{\pi}{2}$



$y = r\sin\theta$, i.e., $r\sin\theta < 1$

is the same $y < 1$ in Cartesian coordinates

#7. $r = 10\cos(2\theta)$, $(r_0, \theta_0) = (5, \pi/6)$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$ $\therefore$ Parametric eq-n of this curve is

$$\begin{cases} x = 10\cos(2\theta)\cos\theta \\ y = 10\cos(2\theta)\sin\theta \end{cases}$$

and $$\begin{cases} x_0 = 5\cos\frac{\pi}{6} = \frac{5\sqrt{3}}{2} \\ y_0 = 5\sin\frac{\pi}{6} = \frac{5}{2} \end{cases}$$

Slope $m = \left.\dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right|_{\theta = \frac{\pi}{6}}$.

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}\left(10\cos(2\theta)\sin\theta\right)}{\frac{d}{d\theta}\left(10\cos(2\theta)\cos\theta\right)} =$$

$$= \frac{-\sin(2\theta)\cdot 2\sin\theta + \cos(2\theta)\cdot\cos\theta}{-\sin(2\theta)\cdot 2\cos\theta - \cos(2\theta)\cdot\sin\theta}$$

$$\therefore m = \left.\frac{dy}{dx}\right|_{\theta = \frac{\pi}{6}} = \frac{-\frac{\sqrt{3}}{2}\cdot 2\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}}{-\frac{\sqrt{3}}{2}\cdot 2\cdot\frac{\sqrt{3}}{2} - \frac{1}{2}\cdot\frac{1}{2}} =$$

$$= \frac{-\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}}{-\frac{3}{2} - \frac{1}{4}} = \boxed{\frac{\sqrt{3}}{7}}$$ Slope

Tangent line:

$$\boxed{y - \frac{5}{2} = \frac{\sqrt{3}}{7}\left(x - \frac{5\sqrt{3}}{2}\right)}$$

Answer

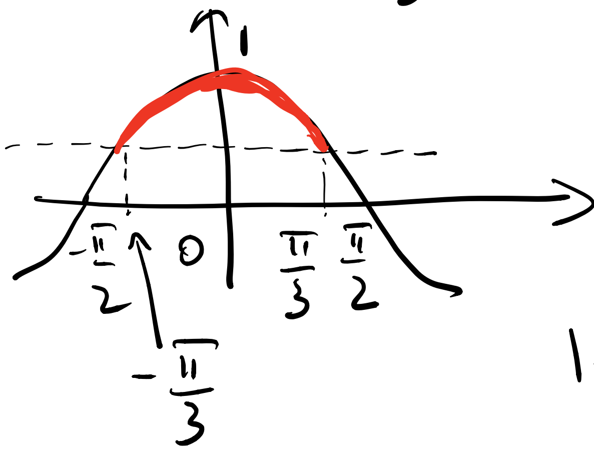
#8. a) We know that, if $f(x) \le g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

Since $(1+x^2)^{1/3} \le (1+x^2)^5 \Leftrightarrow$

$\Leftrightarrow (1+x^2)^{5-\frac{1}{3}} \ge 1 \leftarrow$ true, we have

$$\int_0^1 (1+x^2)^{1/3}dx \le \int_0^1 (1+x^2)^5 dx$$

b) For $-\frac{\pi}{3} \le x \le \frac{\pi}{3}$, $\cos x \geq \frac{1}{2}$



$\therefore \cos^{10} x \geq \frac{1}{2^{10}}$

and so

$1 + 2^{10}\cos^{10} x \geq 2.$

$$\therefore \int_{-\pi/3}^{\pi/3} \left(1 + 2^{10}\cos^{10} x\right) dx \geq \int_{-\pi/3}^{\pi/3} 2\, dx = 2 \cdot \frac{2\pi}{3} =$$

$$= \frac{4\pi}{3}.$$