MATH 1700 Assignment 2 (Sol-ns)

#1 Note: we use the formula

$$\int_{f(x)}^{g(x)} h(u)\,du = \int_{c}^{g(x)} h(u)\,du + \int_{f(x)}^{c} h(u)\,du =$$

$$= \int_{c}^{g(x)} h(u)\,du - \int_{c}^{f(x)} h(u)\,du,$$ and so

$$\frac{d}{dx}\left[\int_{f(x)}^{g(x)} h(u)\,du\right] = h(g(x))\cdot g'(x) - h(f(x))\cdot f'(x)$$

a) $$f(x) = \int_{\tan x}^{3} \frac{2^u}{u^2}\,du$$

$$f'(x) = 0 - \frac{2^{\tan x}}{(\tan x)^2}\cdot(\tan x)' =$$

$$= \boxed{-\frac{2^{\tan x}}{\tan^2 x}\cdot\sec^2 x} = \boxed{-\frac{2^{\tan x}}{\sin^2 x}} = \boxed{-2^{\tan x}\csc^2 x}$$

Answer

b) $g(x) = \int_{1+\sin x}^{x^4} \frac{\tan u}{u}\, du$

$$g'(x) = \frac{\tan(x^4)}{x^4} \cdot (x^4)' - \frac{\tan(1+\sin x)}{1+\sin x}(1+\sin x)' =$$

$$= \boxed{4\,\frac{\tan(x^4)}{x} - \frac{\tan(1+\sin x)}{1+\sin x} \cdot \cos x}$$

Answer

#2a) $\int_0^{\pi/4} \sqrt{\tan x}\,(\sec x)^4\, dx =$

$$= \left\{ \begin{array}{l} u = \tan x \\ du = \sec^2 x\, dx \\ \sec^2 x = 1 + \tan^2 x \end{array} \;\middle|\; \begin{array}{l} x = 0 \to u = 0 \\ x = \frac{\pi}{4} \to u = 1 \end{array} \right\} =$$

$$= \int_0^1 \sqrt{u}\,(1+u^2)\, du = \int_0^1 \left(u^{\frac{1}{2}} + u^{\frac{5}{2}}\right) du =$$

Answer

$$= \left(\frac{2}{3}u^{\frac{3}{2}} + \frac{2}{7}u^{\frac{7}{2}}\right)\Big|_0^1 = \frac{2}{3} + \frac{2}{7} = \boxed{\frac{20}{21}}$$

b) $$\int_1^{\sqrt{e}} \frac{\sin(\pi \ln x)}{2x}\,dx =$$

$$= \left\{ \begin{array}{l|l} u = \pi \ln x & x=1 \to u = \pi \ln 1 = 0 \\ du = \frac{\pi}{x}dx & x=\sqrt{e} \to u = \pi \ln\sqrt{e} = \frac{\pi}{2} \end{array} \right\} =$$

$$= \int_1^{\sqrt{e}} \underbrace{\sin(\pi \ln x)}_{\sin u} \cdot \frac{1}{2\pi} \cdot \underbrace{\frac{\pi}{x}dx}_{du} =$$

$$= \int_0^{\pi/2} \frac{1}{2\pi} \sin u \cdot du = \frac{1}{2\pi}(-\cos u)\Big|_0^{\pi/2} =$$

$$= -\frac{1}{2\pi}\left(\cos\frac{\pi}{2} - \cos 0\right) = -\frac{1}{2\pi}(-1) = \boxed{\frac{1}{2\pi}}$$

Answer

#3 a) $$\int \frac{x^2}{\sqrt{1-x}}\,dx = \left\{ \begin{array}{l} u = 1-x \\ x = 1-u \\ dx = -du \end{array} \right\} =$$

$$= -\int \frac{(1-u)^2}{\sqrt{u}}\,du = -\int \frac{u^2-2u+1}{\sqrt{u}}\,du =$$

$$= -\int \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + u^{-\frac{1}{2}}\right) du =$$

$$= -\frac{2}{5}u^{\frac{5}{2}} + 2\cdot\frac{2}{3}u^{\frac{3}{2}} - 2\cdot u^{\frac{1}{2}} + C =$$

$$= \boxed{-\frac{2}{5}(1-x)^{\frac{5}{2}} + \frac{4}{3}(1-x)^{3/2} - 2(1-x)^{\frac{1}{2}} + C}$$

$$= \boxed{\left[-\frac{2}{5}(1-x)^2 + \frac{4}{3}(1-x) - 2\right]\sqrt{1-x} + C}$$

Answer

Note: you don't have to simplify your answer unless it is explicitly stated that your answers have to be simplified.

b) $\int \cos x \cdot \sin^3(\sin x)\,dx = \left\{\begin{array}{l} u = \sin x \\ du = \cos x\,dx \end{array}\right\} =$

$= \int \sin^3 u \cdot du = \left\{\begin{array}{l} v = \cos u \\ dv = -\sin u\,du \\ \sin^2 u = 1 - \cos^2 u \end{array}\right\} =$

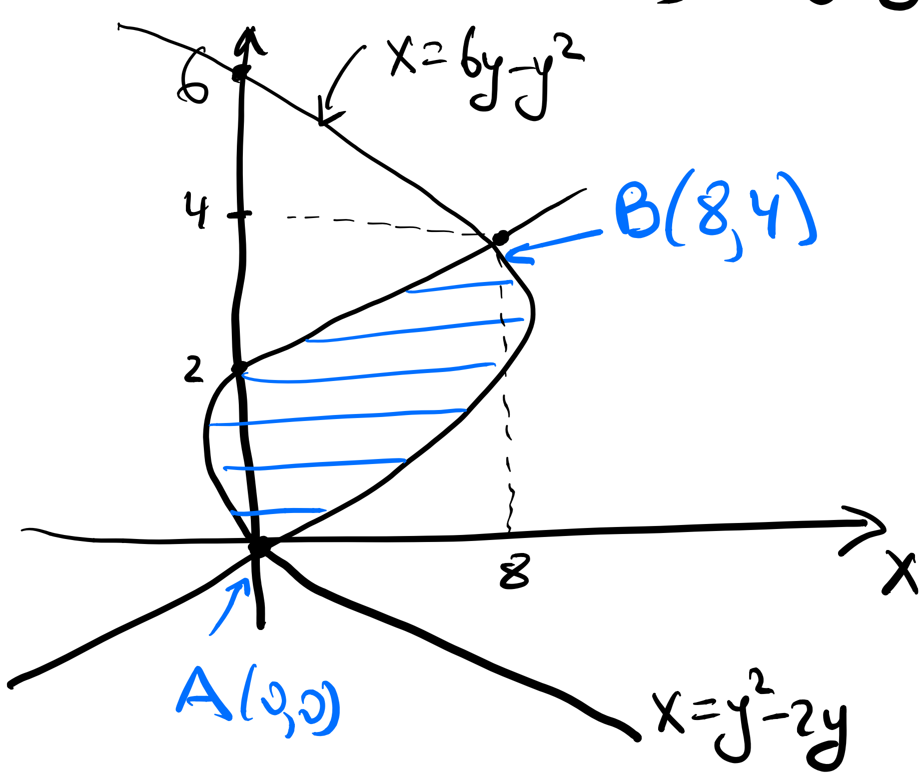
$$= \int \underbrace{\sin^2 u}_{1-v^2} \cdot \underbrace{\sin u \, du}_{-dv} = \int (v^2-1)\,dv =$$

$$= \frac{1}{3}v^3 - v + C = \frac{1}{3}(\cos u)^3 - \cos u + C =$$

$$= \boxed{\frac{1}{3}\left[\cos(\sin x)\right]^3 - \cos(\sin x) + C}$$

Answer

#4. a) Region: between $x = y^2 - 2y$, $x = 6y - y^2$.



Points of intersection:

$$y^2 - 2y = 6y - y^2 \iff 2y^2 = 8y \iff y^2 - 4y = 0$$

$$\iff y = 0 \text{ or } y = 4, \text{ i.e.,}$$

$A(0,0)$ and $B(8,4)$

Hence, the area is

$$\int_0^4 \left[(6y - y^2) - (y^2 - 2y)\right] dy = \int_0^4 (8y - 2y^2)\, dy =$$

$$= \left(4y^2 - \tfrac{2}{3}y^3\right)\Big|_0^4 = 4^3 - \tfrac{2}{3}\cdot 4^3 = \tfrac{1}{3}\cdot 4^3 = \boxed{\frac{64}{3}}$$

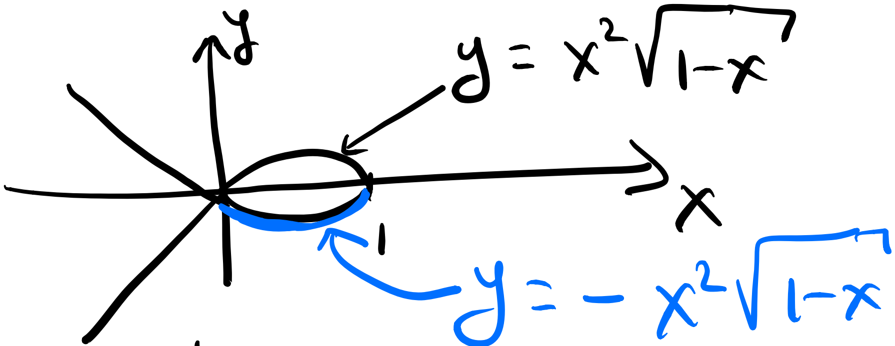
Answer

b) Closed loop of $y^2 = x^4/(1-x)$ with $x \geq 0$.

→ $y = 0 \iff x = 0$ or $x = 1$

→ Also, since $y^2 \geq 0$, we must have $1 - x \geq 0$, i.e., $x \leq 1$.

→ The curve is symmetric about the x-axis, since (x, y) on the curve $\Rightarrow (x, -y)$ is on the curve.



$$\text{Area} = 2\int_0^1 x^2\sqrt{1-x}\,dx =$$

$$= \left\{ \begin{array}{l} u = 1-x \\ x = 1-u \\ dx = -du \end{array} \middle| \begin{array}{l} x=0 \to u=1 \\ x=1 \to u=0 \end{array} \right\} =$$

$$= 2\int_1^0 (1-u)^2\sqrt{u}\,(-1)\,du =$$

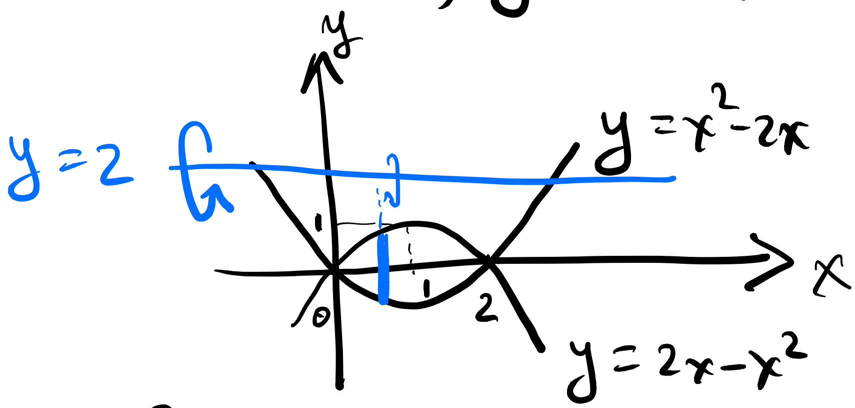
$$= 2\int_0^1 (1-u)^2\sqrt{u}\,du = 2\int_0^1 (u^2-2u+1)\sqrt{u}\,du =$$

$$= 2\int_0^1 \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du =$$

$$= 2\left(\frac{2}{7}u^{\frac{7}{2}} - 2\cdot\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right)\Bigg|_0^1 =$$

$$= 2\left(\frac{2}{7} - \frac{4}{5} + \frac{2}{3}\right) = \boxed{\frac{32}{105}}$$ Answer

\# 5. $y = x^2 - 2x$, $y = 2x - x^2$ about $y = 2$



$$V = \int_0^2 \left[\pi(\text{outer rad})^2 - \pi(\text{inner rad})^2\right] dx =$$

$$= \int_0^2 \left(\pi\left[2 - (x^2 - 2x)\right]^2 - \pi\left[2 - (2x - x^2)\right]^2\right) dx =$$

Answer for part (a)

$$= \boxed{\pi \int_0^2 \left[(2 - x^2 + 2x)^2 - (2 + x^2 - 2x)^2\right] dx} =$$

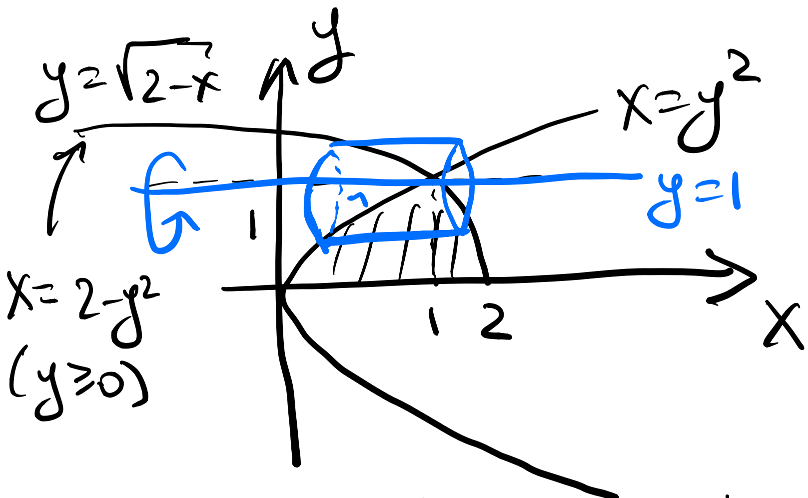
$$= \pi \int_0^2 (2 \cancel{-x^2} \cancel{+2x} + 2 \cancel{+x^2} \cancel{-2x})(\cancel{2} - x^2 + 2x \cancel{-2} - x^2 + 2x) dx$$

$$= \pi \int_0^2 4 \cdot (4x - 2x^2) dx = 8\pi \int_0^2 (2x - x^2) dx =$$

$$= 8\pi \left(x^2 - \frac{1}{3}x^3\right)\Big|_0^2 = 8\pi\left(4 - \frac{8}{3}\right) = \boxed{\frac{32\pi}{3}}$$

Answer

#6. $x=y^2, y=\sqrt{2-x}, y=0$ about $y=1$



Find pt. of intersection:

$x=y^2=(\sqrt{2-x})^2 \Rightarrow x=2-x \Rightarrow x=1$

$\therefore (1,1)$ is the pt. of int. of $x=y^2$ and $y=\sqrt{2-x}$

a) $$V=\int_0^1 2\pi(1-y)\cdot\left(\underset{\text{in terms of } y}{\text{Height of shell}}\right)dy$$

$$=\int_0^1 2\pi(1-y)(2-y^2-y^2)\,dy=$$

$$=\boxed{4\pi\int_0^1(1-y)(1-y^2)\,dy}=$$

Answer for (a)

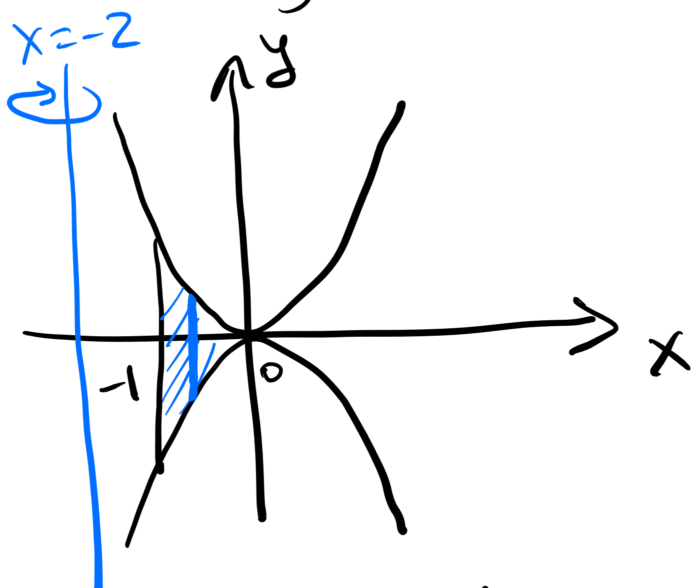
6)

$$= 4\pi \int_0^1 (1-y-y^2+y^3)\,dy =$$

$$= 4\pi \left(y - \frac{1}{2}y^2 - \frac{1}{3}y^3 + \frac{1}{4}y^4\right)\Big|_0^1 =$$

$$= 4\pi\left(1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4}\right) = \boxed{\frac{5\pi}{3}}$$ Answer

#7. $y = x^2$, $y = -x^2$, $x = -1$ about $x = -2$



We use shells method:

$$V = \int_{-1}^{0} 2\pi(x-(-2))(x^2-(-x^2))\,dx =$$

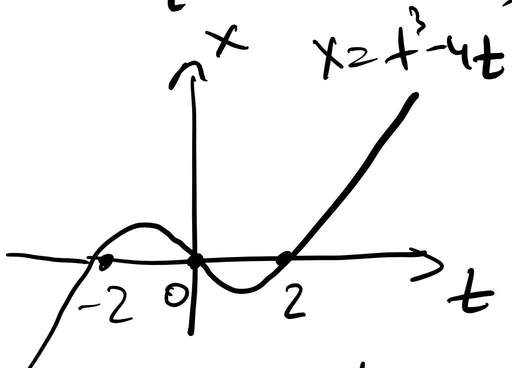
$$= \boxed{\int_{-1}^{0} 2\pi (x+2)\cdot 2x^2\,dx} =$$ Answer for (a)

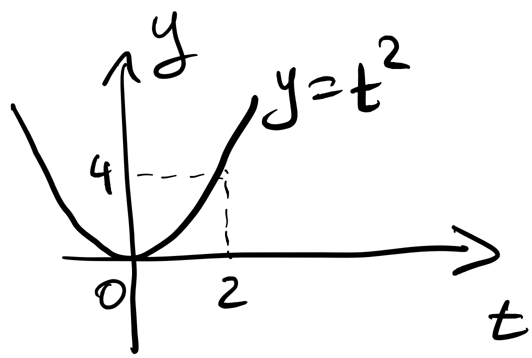
$$= 4\pi \int_{-1}^{0} (x^3+2x^2)\,dx = 4\pi\left(\frac{1}{4}x^4 + \frac{2}{3}x^3\right)\Big|_{-1}^{0} =$$

$$= 4\pi\left(-\frac{1}{4} - \frac{2}{3}(-1)^3\right) = 4\pi\left(\frac{2}{3} - \frac{1}{4}\right) =$$

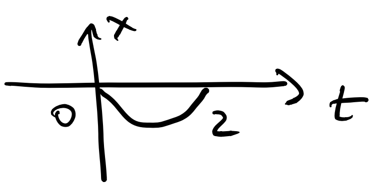
$$= \boxed{\frac{5\pi}{3}}$$ Answer for (b)

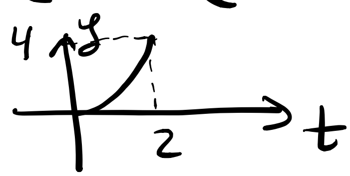
#8. $\begin{cases} x = t^3 - 4t \\ y = t^2 \end{cases}, \quad 0 \le t \le 2$ and y-axis



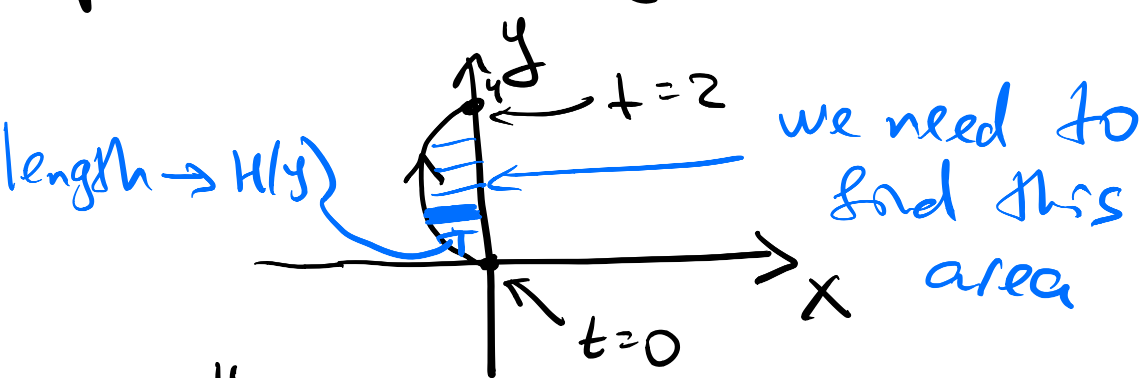


For $0 \le t \le 2$, we have





We use the above to sketch this parametric curve



$$A = \int_0^4 H(y)\,dy = \int_0^4 (0-x)\,dy =$$

$$= -\int_0^4 x\,dy = \left\{ \begin{array}{l} x = t^3 - 4t \\ y = t^2 \\ dy = 2t\,dt \end{array} \right\} =$$

$$= -\int_0^2 (t^3 - 4t)\cdot 2t\,dt =$$

$$= \int_0^2 (8t^2 - 2t^4)\,dt = \left(\frac{8}{3}t^3 - \frac{2}{5}t^5\right)\Big|_0^2 =$$

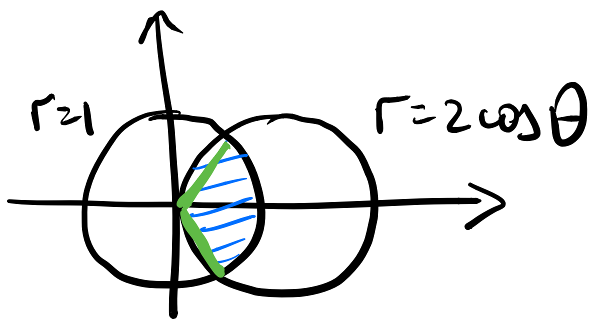
$$= \frac{8}{3}\cdot 2^3 - \frac{2}{5}\cdot 2^5 = 64\left(\frac{1}{3} - \frac{1}{5}\right) = \boxed{\frac{128}{15}}$$ Answer

#9. $r=1$ and $r=2\cos\theta$

Note: $r=2\cos\theta \Rightarrow \underbrace{r^2}_{x^2+y^2}=\underbrace{2r\cos\theta}_{2x}$

$\Rightarrow x^2+y^2=2x \Rightarrow (x-1)^2+y^2=1$, i.e.,

$r=2\cos\theta$ is the circle of radius 1 centered at $(1,0)$.



Warning: the green lines show where the boundary changes from one polar curve to the other.

$$\begin{cases} r=1 \\ r=2\cos\theta \end{cases} \Rightarrow \cos\theta=\frac{1}{2} \Rightarrow \theta=\pm\frac{\pi}{3}$$

$\therefore\ \theta_{1,2}=\pm\frac{\pi}{3}.$

Method 1: $A = A_1 + A_2 + A_3$

$\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$

$-\frac{\pi}{2} \le \theta \le -\frac{\pi}{3}$ $\quad$ $-\frac{\pi}{3} \le \theta \le \frac{\pi}{3}$

$$A_1 = \int_{-\frac{\pi}{2}}^{-\frac{\pi}{3}} \frac{1}{2}\left[2\cos\theta\right]^2 d\theta$$

$$A_3 = \int_{\frac{\pi}{3}}^{\pi/2} \frac{1}{2}\left[2\cos\theta\right]^2 d\theta \quad (= A_1)$$

$$A_2 = \int_{-\frac{\pi}{3}}^{\pi/3} \frac{1}{2}\left[1\right]^2 d\theta = \frac{\pi}{3}$$

Hence, $A_1 = A_3 = 2\int_{\pi/3}^{\pi/2} \cos^2\theta \, d\theta =$

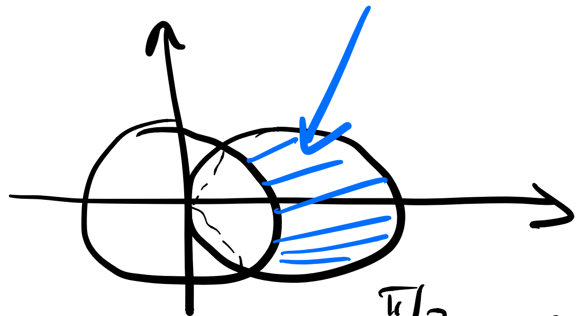
$$= 2\int_{\pi/3}^{\pi/2} \frac{1+\cos 2\theta}{2} d\theta = \left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_{\pi/3}^{\pi/2} =$$

$$= \frac{\pi}{2} + \frac{1}{2}\sin\pi - \frac{\pi}{3} - \frac{1}{2}\sin\frac{2\pi}{3} = \frac{\pi}{6} - \frac{\sqrt{3}}{4}$$

$\therefore A = 2\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) + \frac{\pi}{3} = \boxed{\frac{2\pi}{3} - \frac{\sqrt{3}}{2}}.$

Answer

Method 2: A = Area of the unit circle — [Area inside $r = 2\cos\theta$ and outside $r = 1$]



$$\therefore A = \pi - \int_{-\frac{\pi}{3}}^{\pi/3} \frac{1}{2}\left(\left[2\cos\theta\right]^2 - 1^2\right) d\theta$$

$$= \pi - \int_{-\frac{\pi}{3}}^{\pi/3} \left(2\cos^2\theta - \frac{1}{2}\right) d\theta =$$

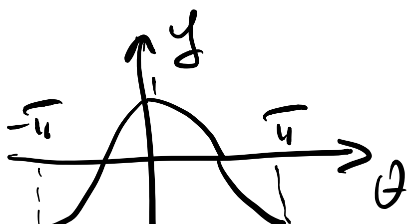
$$= \pi - \int_{-\frac{\pi}{3}}^{\pi/3} \left(\cos 2\theta + 1 - \frac{1}{2}\right) d\theta =$$

$$= \pi - \left(\frac{1}{2}\sin 2\theta + \frac{1}{2}\theta\right)\Big|_{-\frac{\pi}{3}}^{\pi/3} =$$

$$= \pi - \left(\sin\frac{2\pi}{3} + \frac{\pi}{3}\right) = \boxed{\frac{2\pi}{3} - \frac{\sqrt{3}}{2}}$$

Answer

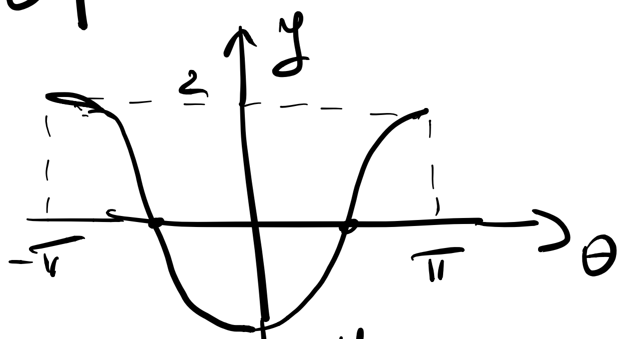
#10. $r = |1 - 2\cos\theta|$



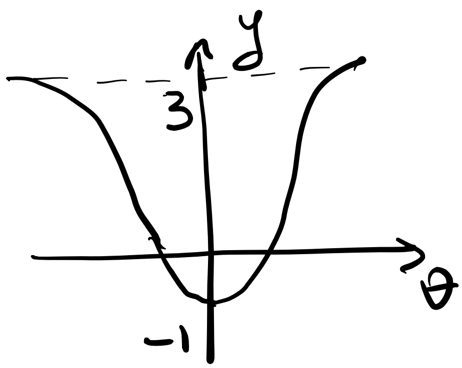
$y = \cos\theta$

$-\pi \le \theta \le \pi$

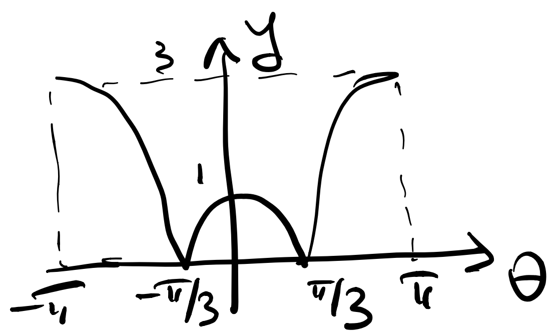
$\Rightarrow$



-2 $\quad y = -2\cos\theta$



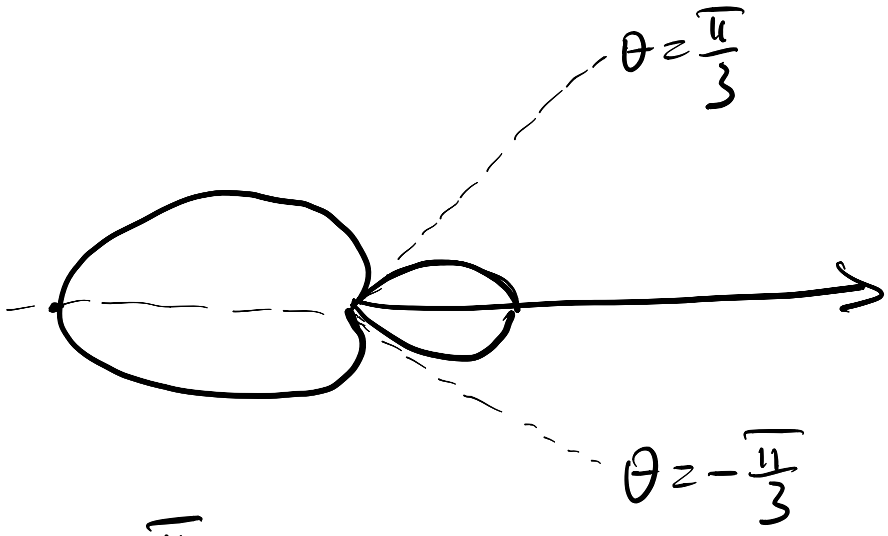
$y = 1 - 2\cos\theta$



$y = |1 - 2\cos\theta|$

$$1 - 2\cos\theta = 0 \iff \cos\theta = \frac{1}{2} \iff \theta = \pm\frac{\pi}{3} \text{ for } -\pi \le \theta \le \pi$$

We now use the last graph to sketch this polar curve



$$A = \int_{-\pi}^{\pi} \frac{1}{2}\left[|1-2\cos\theta|\right]^2 d\theta = \left\{\begin{matrix}\text{use} \\ \text{symmetry}\end{matrix}\right\}$$

$$= \int_0^{\pi} (1-2\cos\theta)^2 d\theta = \int_0^{\pi} (1-4\cos\theta+4\cos^2\theta)\,d\theta$$

$$= \int_0^{\pi} \left(1-4\cos\theta+2(\cos 2\theta+1)\right) d\theta =$$

$$= \int_0^{\pi} (3-4\cos\theta+2\cos 2\theta)\,d\theta =$$

$$= (3\theta-4\sin\theta+\sin 2\theta)\Big|_0^{\pi} = \boxed{3\pi}$$

Answer