MATH 1700 Assignment 3 (Solutions)

#1. a) $\sin^{-1}(\sin \frac{3\pi}{4}) = \sin^{-1}(\frac{1}{\sqrt{2}}) = \boxed{\frac{\pi}{4}}$

(recall: range of $\sin^{-1}x$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$)

b) $\sin(\sin^{-1} \frac{3\pi}{4})$ $\boxed{\text{d.n.e.}}$ since domain of $\sin^{-1}x$ is $[-1,1]$, and $\frac{3\pi}{4} \notin [-1,1]$

c) $\cos^{-1}(\cos(-\frac{\pi}{4})) = \cos^{-1}(\frac{1}{\sqrt{2}}) = \boxed{\frac{\pi}{4}}$

(range of $\cos^{-1}x$ is $[0,\pi]$).

d) $\tan(\tan^{-1} 100) = \boxed{100}$

(cancellation eq-n for tangent).

#2. $\sin^{-1}(xy) = 3x+2y$

Differentiate wrt x:

$$\frac{d}{dx}\left[\sin^{-1}(xy)\right] = \frac{d}{dx}(3x+2y)$$

$$\Leftrightarrow \frac{1}{\sqrt{1-x^2y^2}} \frac{d}{dx}(xy) = 3 + 2\frac{dy}{dx}$$

$$\Leftrightarrow \frac{1}{\sqrt{1-x^2y^2}}\left(y+x\frac{dy}{dx}\right)=3+2\frac{dy}{dx}$$

Solve for $\frac{dy}{dx}$:

$$y+x\frac{dy}{dx}=3\sqrt{1-x^2y^2}+2\sqrt{1-x^2y^2}\,\frac{dy}{dx}$$

$$\Leftrightarrow \boxed{\frac{dy}{dx}=\frac{3\sqrt{1-x^2y^2}-y}{x-2\sqrt{1-x^2y^2}}}$$

by parts

#3. $\int \ln(x^2+1)\,dx = \int x'\cdot \ln(x^2+1)\,dx =$

$= x\ln(x^2+1) - \int x\cdot\frac{1}{x^2+1}\cdot 2x\,dx =$

$= x\ln(x^2+1) - 2\int\left(1-\frac{1}{x^2+1}\right)dx =$

$= \boxed{x\ln(x^2+1) - 2x + 2\tan^{-1}x + C,\quad C\in\mathbb{R}}$

integration by parts

#4. $\int \cos^{-1}x \, dx = \int x' \cdot \cos^{-1}x \, dx =$

$= x\cos^{-1}x - \int x \cdot \frac{1}{\sqrt{1-x^2}}(-1)\,dx =$

$= x\cos^{-1}x + \int \frac{x}{\sqrt{1-x^2}}\,dx$

$= \boxed{x\cos^{-1}x - \sqrt{1-x^2} + C, \ C \in \mathbb{R}}$

#5. $\int \tan^{-1}\sqrt{x}\,dx = \left\{ \begin{array}{l} u=\sqrt{x},\ x=u^2 \\ dx = 2u\,du \end{array} \right\}$

$= \int 2u \tan^{-1}u \, du = \int (u^2)' \tan^{-1}u \, du =$

$= u^2\tan^{-1}u - \int u^2 \cdot \frac{1}{1+u^2}\,du =$

$= u^2\tan^{-1}u - \int \left(1 - \frac{1}{1+u^2}\right)du =$

$= u^2\tan^{-1}u - u + \tan^{-1}u + C =$

$$= \boxed{(x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C, \quad C \in \mathbb{R}}$$

#6. $$\int \frac{\tan^3 x}{\sec^4 x}\,dx = \int \tan^3 x \cdot \cos^4 x\,dx =$$

$$= \int \sin^3 x \cdot \cos x\,dx = \left\{\begin{array}{l} u = \sin x \\ du = \cos x\,dx \end{array}\right\} =$$

$$= \int u^3\,du = \frac{1}{4}u^4 + C = \boxed{\frac{1}{4}\sin^4 x + C,}$$

$C \in \mathbb{R}$

#7. $$\int \frac{1}{\cos^4 x}\,dx = \int \sec^4 x\,dx =$$

$$= \left\{\begin{array}{l} u = \tan x \\ du = \sec^2 x\,dx \\ 1 + \tan^2 x = \sec^2 x \\ \sec^2 x = 1 + u^2 \end{array}\right\} = \int (1+u^2)\,du =$$

$$= u + \frac{1}{3}u^3 + C = \boxed{\tan x + \frac{1}{3}\tan^3 x + C,}$$

$C \in \mathbb{R}$

#8. $\int\sqrt{-x^2+6x}\,dx = \int\sqrt{9-(x-3)^2}\,dx =$

$$= \left\{ \begin{array}{l} x-3 = 3\sin\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ dx = 3\cos\theta\, d\theta \\ \sqrt{9-(x-3)^2} = \sqrt{9-9\sin^2\theta} = 3|\cos\theta| = 3\cos\theta, \\ \quad \text{since } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \end{array} \right\}$$

$$= \int 3\cos\theta \cdot 3\cos\theta\, d\theta = 9\int \cos^2\theta\, d\theta =$$

$$= 9\int \frac{1+\cos 2\theta}{2}\, d\theta = \frac{9}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C =$$

$$= \frac{9}{2}(\theta + \sin\theta \cdot \cos\theta) + C =$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x-3}{3}\right) + \frac{9}{2} \cdot \frac{x-3}{3} \cdot \frac{\sqrt{9-(x-3)^2}}{3} + C =$$

$$= \boxed{\frac{9}{2}\sin^{-1}\left(\frac{x-3}{3}\right) + \frac{1}{2}(x-3)\sqrt{6x-x^2} + C,}$$

$C \in \mathbb{R}$

#9. $\int \frac{x^2}{x^2+3x+2}\,dx = \int\left(1-\frac{3x+2}{x^2+3x+2}\right)dx =$

$$= x - \int \frac{3x+2}{(x+1)(x+2)}\,dx =$$

$$\frac{3x+2}{(x+1)(x+2)} = \frac{a}{x+1} + \frac{b}{x+2}$$

$$3x+2 = a(x+2) + b(x+1)$$

$$x=-1:\ -1 = a\cdot 1 \Rightarrow a=-1$$

$$x=-2:\ -4 = b(-1) \Rightarrow b=4$$

$$= x - \int\left((-1)\frac{1}{x+1} + \frac{4}{x+2}\right)dx =$$

$$= \boxed{x - 4\ln|x+2| + \ln|x+1| + C,\ C\in\mathbb{R}}$$

#10. $\int \frac{\sin x}{\cos x(1+\cos^2 x)}\,dx = \left\{ \begin{array}{l} u=\cos x \\ du=-\sin x\,dx \end{array} \right\}$

$= -\int \frac{du}{u(1+u^2)} \; (=)$

$$\left\{ \begin{array}{l} \frac{1}{u(1+u^2)} = \frac{a}{u} + \frac{bu+c}{u^2+1} \\ 1 = a(u^2+1) + (bu+c)u \iff \\ 1 = u^2(a+b) + cu + a \\ \therefore\ a=1,\ c=0,\ a+b=0 \Rightarrow b=-1 \end{array} \right\}$$

$(=) -\int \left(\frac{1}{u} - \frac{u}{1+u^2}\right) du = \int \left(\frac{u}{1+u^2} - \frac{1}{u}\right) du =$

$= \frac{1}{2}\ln(1+u^2) - \ln|u| + C =$

$= \boxed{\frac{1}{2}\ln(1+\cos^2 x) - \ln|\cos x| + C,\ C \in \mathbb{R}}$

#11. $\int \sin x \cdot \cos 2x \, dx =$

$$= \frac{1}{2} \int \left[\sin(x+2x) + \sin(x-2x) \right] dx =$$

$$= \frac{1}{2} \int \left[\sin 3x - \sin x \right] dx =$$

$$= \frac{1}{2} \left[-\frac{1}{3} \cos 3x + \cos x \right] + C =$$

$$= \boxed{-\frac{1}{6} \cos 3x + \frac{1}{2} \cos x + C, \ C \in \mathbb{R}.}$$

#12. $\int \frac{2-\sqrt{x}}{\sqrt{x}+5} \, dx = \left\{ \begin{array}{l} u = \sqrt{x}+5 \\ x = (u-5)^2 \\ dx = 2(u-5)\,du \end{array} \right\}$

$$= \int \frac{2-(u-5)}{u} \cdot 2(u-5)\,du =$$

$$= 2 \int \frac{(7-u)(u-5)}{u} \, du = 2 \int \frac{-u^2+12u-35}{u} \, du =$$

$$= 2 \int \left(-u + 12 - \frac{35}{u} \right) du =$$

$$= 2\left[-\frac{1}{2}u^2 + 12u - 35\ln|u|\right] + C =$$

$$= -(\sqrt{x}+5)^2 + 24(\sqrt{x}+5) - 70\ln(\sqrt{x}+5) + C$$

$$= \boxed{-x + 14\sqrt{x} - 70\ln(\sqrt{x}+5) + \tilde{C},\ \tilde{C} \in \mathbb{R}}$$

#13. $I = \int \sin(\ln x)\,dx = \int x' \cdot \sin(\ln x)\,dx$

$$= x\sin(\ln x) - \int \cancel{x} \cdot \cos(\ln x) \cdot \frac{1}{\cancel{x}}\,dx =$$

$$= x\sin(\ln x) - \int x' \cdot \cos(\ln x)\,dx =$$

$= \{$ use Integration by parts again $\}$

$$= x\sin(\ln x) - x \cdot \cos(\ln x) +$$

$$+ \int \cancel{x} \cdot (-\sin(\ln x)) \cdot \frac{1}{\cancel{x}}\,dx$$

Hence,

$$I = x\sin(\ln x) - x \cdot \cos(\ln x) - I$$

$$\therefore \boxed{I = \frac{1}{2}x\left[\sin(\ln x) - \cos(\ln x)\right] + C,\ C \in \mathbb{R}}$$