Assignment 3 Solutions

1-3.

Outcome	Probability	Sum (for Question 3)
11	2/12	2
12	2/12	3
13	2/12	4
21	2/12	3
23	1/12	5
31	2/12	4
32	1/12	5

An example of how to obtain the probabilities follows:

$$\begin{aligned}
P(11) &= P(1 \text{ on 1st selection} \cap 1 \text{ on 2nd selection}) \\
&= P(1 \text{ on 1st selection}) \times P(1 \text{ on 2nd selection} | 1 \text{ on 1st selection}) \\
&= (2/4) \times (1/3) = 2/12
\end{aligned}$$

1. There are 7 outcomes in the sample space, as seen above.

2.

$$\begin{aligned}
P(X = 2) &= P(11) = 2/12 \\
P(X = 3) &= P(12) + P(21) = 4/12 \\
P(X = 4) &= P(13) + P(31) = 4/12 \\
P(X = 5) &= P(23) + P(32) = 2/12
\end{aligned}$$

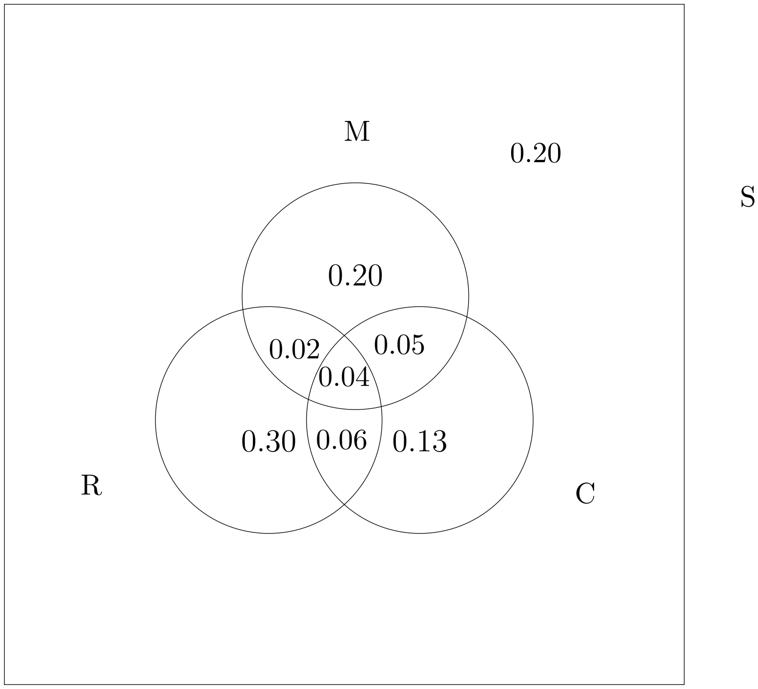
3.

$$P(X = 3 | \text{ 1st selection not a "3"}) = \frac{P(X = 3 \cap \text{ 1st selection not a "3"})}{P(\text{1st selection not a "3"})}$$

There are 2 outcomes that satisfy $\{X = 3\}$, namely 12 and 21. There are 5 outcomes that satisfy {1st selection not a "3"}, namely 11, 12, 13, 21, and 23. The intersection of these outcomes $P(X = 3 \cap$ 1st selection not a "3") results in only 12 and 21. Therefore,

$$\begin{aligned}
P(X = 3 | \text{ 1st selection not a "3"}) &= \frac{P(X = 3 \cap \text{ 1st selection not a "3"})}{P(\text{1st selection not a "3"})} \\
&= \frac{P(12) + P(21)}{P(11) + P(12) + P(13) + P(21) + P(23)} \\
&= \frac{4/12}{9/12} \\
&= 4/9
\end{aligned}$$

4-6. Let R denote the event that the selected person reads *Reader's Digest*, let M denote the event that the selected person reads *Maclean's*, and let C denote the event that the selected person reads *Canadian Geographic*.



4.

$$
\begin{aligned}
P(M \cup C) &= P(M) + P(C) - P(M \cap C) \\
0.50 &= 0.31 + P(C) - 0.09 \\
\Rightarrow P(C) &= 0.28
\end{aligned}
$$

5.

$$P(M|R^c) = \frac{P(M \cap R^c)}{P(R^c)} = \frac{0.25}{1 - 0.42} = 0.43$$

6.

$$P(M|R \cap C) = \frac{P(M \cap R \cap C)}{P(R \cap C)} = \frac{0.04}{0.10} = 0.40$$

7. If the first two selected cards are not spades, there are 50 cards left in the deck and 13 of those remaining cards are spades, so the answer is $\frac{13}{50}$.

8. Method 1: $P(A) = \frac{1}{13}$, $P(H) = \frac{1}{4}$, and $P(A \cap H) = \frac{1}{52}$. Since $P(A \cap H) = P(A)P(H)$, A and H are independent.

   Method 2: $P(H|A) = \frac{1}{4} = P(H)$, so A and H are independent.

9. $E(X) = 1(0.2) + 2(0.3) + 3(0.5) = 2.3$

10. $Var(X) = (1-2.3)^2(0.2) + (2-2.3)^2(0.3) + (3-2.3)^2(0.5) = 0.61$

11. $X_m \sim N(530, 70) \quad E(X_m - X_v) = 530 - 475 = 55$
    $X_v \sim N(475, 75) \quad Var(X_m - X_v) = 70^2 + 75^2 = 10525$

$$\begin{aligned} P(X_m > X_v) = P(X_m - X_v > 0) &= P\left(Z > \frac{0-55}{\sqrt{10525}}\right) \\ &= P(Z > -0.54) = 1 - P(Z \le -0.54) \\ &= 1 - 0.2946 = 0.7054 \end{aligned}$$

12. $X =$ quiz scores $\sim N(70, 10)$
    $P(X > 85) = P\left(Z > \dfrac{85-70}{10}\right) = P(Z > 1.50) = 0.0668$
    $Y = \#$ of students with a quiz score above 85 $\sim \text{Bin}(10, 0.0668)$
    $P(Y \ge 1) = 1 - P(Y = 0) = 1 - \binom{10}{0}(0.0668)^0(0.9332)^{10} = 0.4991$

13. The circular area is $\pi(3)^2 = 28.2743$.
    $\lambda = (0.2)(28.2743) = 5.6549$
    $P(X = 5) = \dfrac{e^{-5.6549}(5.6549)^5}{5!} = 0.1687$

14. Since this is a two-sided test with $\alpha = 0.10$, we must find the values $-z^*$ and z^* such that $P(Z < -z^*) = 0.05$ and $P(Z > z^*) = 0.05$. From Table 3, we find that $z^* = 1.645$. We will reject H_0
    if $Z = \dfrac{\hat{p} - p_0}{\sqrt{\dfrac{p_0(1-p_0)}{n}}} \le -z^*$ or if $Z = \dfrac{\hat{p} - p_0}{\sqrt{\dfrac{p_0(1-p_0)}{n}}} \ge z^*$
    $\Rightarrow \hat{p} \le p_0 - z^*\sqrt{\dfrac{p_0(1-p_0)}{n}} = 0.10 - 1.645\sqrt{\dfrac{0.10(0.90)}{200}} = 0.0651$
    or $\hat{p} \ge p_0 + z^*\sqrt{\dfrac{p_0(1-p_0)}{n}} = 0.10 + 1.645\sqrt{\dfrac{0.10(0.90)}{200}} = 0.1349$

15. For a 94% confidence interval, $z^* = 1.88$ from Table 2.
    $\hat{p}_1 = \dfrac{28}{400} = 0.07 \qquad \hat{p}_2 = \dfrac{15}{300} = 0.05$
    94% CI: $(0.07 - 0.05) \pm 1.88\sqrt{\dfrac{(0.07)(0.93)}{400} + \dfrac{(0.05)(0.95)}{300}}$
    $\Rightarrow 0.02 \pm 1.88(0.017919)$
    $\Rightarrow 0.02 \pm 0.034$
    $\Rightarrow (-0.014, 0.054)$

16. H_0: the distribution of final grades is homogeneous across the courses

    H_a: the distribution of final grades is not homogeneous across the courses

    Equivalently, H_0: $p_B = p_C = p_P$ vs. H_a: at least one p_i differs

17. $df = (r-1)(c-1) = 7 \times 2 = 14$

    Reject H_0 if $\chi^2 \geq 29.14$

18. Missing expected count: $\dfrac{(39)(165)}{526} = 12.23$

    Missing cell chi-square: $\dfrac{(18-12.23)^2}{12.23} = 2.72$

    $\chi^2 = \sum \dfrac{(O-E)^2}{E} = 14.49 + 2.72 = 17.21$

19. Since $17.21 < 29.14$, we fail to reject H_0. We have insufficient evidence to conclude that the distribution of final grades is not homogeneous across the courses.

20. Let $\alpha = 0.05$.

    We are testing the hypotheses

    H_0: The number of female contestants follows a binomial distribution with parameter $p = 0.25$.
    H_a: The number of female contestants does not follow a binomial distribution with parameter $p = 0.25$.

    We first find probabilities for a binomial random variable X with parameters $n = 3$ and $p = 0.25$:

    $P(X=0) = \binom{3}{0}(0.25)^0(0.75)^3 = 0.4219$

    $P(X=1) = \binom{3}{1}(0.25)^1(0.75)^2 = 0.4219$

    $P(X=2) = \binom{3}{2}(0.25)^2(0.75)^1 = 0.1406$

    $P(X=3) = \binom{3}{3}(0.25)^3(0.75)^0 = 0.0156$

    The expected cell counts are calculated as follows:

    For $X=0$, $E = 100(P(X=0)) = 100(0.4219) = 42.19$
    For $X=1$, $E = 100(P(X=1)) = 100(0.4219) = 42.19$
    For $X=2$, $E = 100(P(X=2)) = 100(0.1406) = 14.06$

For $X = 3$, $E = 100(P(X = 3)) = 100(0.0156) = 1.56$

Since the cell for three female contestants has an expected count less than five, we must merge the cells for $X = 2$ and $X = 3$. The new cell for $X \geq 2$ will have an observed count of $15 + 5 = 20$ and an expected count of $14.06 + 1.56 = 15.62$. The final table is shown below:

# of females	0	1	≥ 2
# of episodes	45	35	20
Expected	42.19	42.19	15.62

We calculate the cell chi-square values. For example, the cell chi-square value for 0 female contestants is

$$\frac{(O-E)^2}{E} = \frac{(45 - 42.19)^2}{42.19} = 0.187$$

Other cell chi-square values are calculated similarly and are shown beneath the expected cell counts in the table below:

# of females	0	1	≥ 2
# of episodes	45	35	20
Expected	42.19	42.19	15.62
Cell Chi-Square	0.187	1.23	1.23

The test statistic is

$$\chi^2 = 0.187 + 1.23 + 1.23 = 2.65$$

Under the null hypothesis, this test statistic has a chi-square distribution with degrees of freedom equal to

$$\text{\# of cells } -1 = 3 - 1 = 2$$

Note that we do not have to subtract any additional degrees of freedom, because we didn't have to estimate the values of any parameters.

Using the critical value method, we will reject H_0 if $\chi^2 \geq 5.99$, where $\chi^{2*} = 5.99$ is the upper 0.05 critical value for the chi-square distribution with 2 degrees of freedom.

Since $\chi^2 < 5.99$, we fail to reject H_0. At the 5% level of significance, we have insufficient evidence to conclude that the number of female contestants per episode does not have a binomial distribution with parameter $p = 0.25$.

21. The estimated value of the parameter p is

$$\hat{p} = \frac{\text{number of female contestants in 100 episodes}}{\text{total number of contestants in 100 episodes}}$$

$$= \frac{45(0) + 35(1) + 15(2) + 5(3)}{3(100)} = \frac{80}{300} = 0.27$$