

If $f(x) = |x-6|$ prove $f(x)$ is not differentiable at $x=6$.

ANSWER

$f(x)$ is differentiable at $x=a$ if and only if $f'(a)$ exists where

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

Here, $f(x) = |x-6|$ and $a=6$, so we want to find $f'(6)$

$$\begin{aligned} f'(6) &= \lim_{x \rightarrow 6} \frac{f(x) - f(6)}{x - 6} \\ &= \lim_{x \rightarrow 6} \frac{|x-6| - |0|}{x-6} \\ &= \lim_{x \rightarrow 6} \frac{|x-6|}{x-6} \end{aligned}$$

Use one-sided limits

$$\lim_{x \rightarrow 6^-} \frac{|x-6|}{x-6} = \lim_{x \rightarrow 6^-} \frac{-(x-6)}{x-6} = -1$$

because $|x-6| \rightarrow |⊖| = -(x-6)$

$$\lim_{x \rightarrow 6^+} \frac{|x-6|}{x-6} = \lim_{x \rightarrow 6^+} \frac{x-6}{x-6} = 1$$

because $|x-6| \rightarrow |⊕| = (x-6)$

$\lim_{x \rightarrow 6} \frac{|x-6|}{x-6}$ does not exist because

$$\lim_{x \rightarrow 6^-} \frac{|x-6|}{x-6} \neq \lim_{x \rightarrow 6^+} \frac{|x-6|}{x-6}$$

$$(-1 \neq 1)$$

$f(x)$ is not differentiable at $x=6$.