Grant's Tutoring

INTRO CALCULUS

(DIFFERENTIATION and APPLICATIONS)

Volume 2 of 2

September 2011 edition

This volume covers the topics taught <u>after</u> your midterm exam.

Learn What You Need to Know

Know What You Need to Learn

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HOW TO USE THIS BOOK

I have broken the course up into lessons. Study each lesson until you can do all of my lecture problems from start to finish without any help. Then do the Practise Problems for that lesson. If you are able to solve all the Practise Problems I have given you, then you should have nothing to fear about your Midterm or Final Exam.

I have presented the course in what I consider to be the most logical order. Although my books are designed to follow the course syllabus, it is possible your prof will teach the course in a different order or omit a topic. It is also possible he/she will introduce a topic I do not cover. **Make sure you are attending your class regularly! Stay current with the material, and be aware of what topics are on your exam. Never forget, it is your prof that decides what will be on the exam, so pay attention.**

If you have any questions or difficulties while studying this book, or if you believe you have found a mistake, do not hesitate to contact me. My phone number and website are noted at the bottom of every page in this book. "Grant's Tutoring" is also in the phone book. **I welcome your input and questions.**

Wishing you much success,

Grant Skene

Owner of Grant's Tutoring

Formulas and Definitions to Memorize

The Definition of Continuity: $f(x)$ is continuous at $x=a$ if and only if $\lim\limits_{x \to a} f(x) = f(a)$.

The Definition of Derivative: $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

The Power Rule: $\left(x^n\right)' = nx^{n-1}$

The Product Rule: $(f \cdot g)' = f'g + fg'$

The Quotient Rule: $\left(\dfrac{T}{B}\right)' = \dfrac{T'B - TB'}{B^2}$

The Chain Rule: $\left(f(u)\right)' = f'(u) \cdot u'$

The Chain Rule Version of Power Rule: $\left(u^n\right)' = nu^{n-1} \cdot u'$

Derivatives of Trigonometric Functions:

$(\sin u)' = \cos u \cdot u'$ $\quad (\tan u)' = \sec^2 u \cdot u'$ $\quad (\sec u)' = \sec u \tan u \cdot u'$

$(\cos u)' = -\sin u \cdot u'$ $\quad (\cot u)' = -\csc^2 u \cdot u'$ $\quad (\csc u)' = -\csc u \cot u \cdot u'$

Derivatives of Exponential and Logarithmic Functions:

$\left(e^u\right)' = e^u \cdot u'$ $\quad (\ln u)' = \dfrac{u'}{u}$

$\left(a^u\right)' = a^u \cdot u' \cdot \ln a$ $\quad (\log_a u)' = \dfrac{u'}{(u)\ln a}$

Derivative of an Inverse Function: $\left(f^{-1}\right)'(a) = \dfrac{1}{f'\left(f^{-1}(a)\right)}$

Fundamental Theorem of Calculus: $\left(\int_a^u f(t)\, dt\right)' = f(u) \cdot u'$

Antiderivative Formulas:

$\int K\, dx = Kx + C$

$\int x^n\, dx = \frac{1}{n+1} x^{n+1} + C$

$\int \frac{1}{x}\, dx = \ln|x| + C$

$\int e^x\, dx = e^x + C$, so $\int e^{ax}\, dx = \frac{1}{a} e^{ax} + C$

e.g. $\int e^{5x}\, dx = \frac{1}{5} e^{5x} + C$

$\int a^x\, dx = \frac{a^x}{\ln a} + C$

$\int \sin x\, dx = -\cos x + C$, so $\int \sin ax\, dx = -\frac{1}{a}\cos ax + C$

e.g. $\int \sin(3x)\, dx = -\frac{1}{3}\cos(3x) + C$

$\int \cos x\, dx = \sin x + C$, so $\int \cos ax\, dx = \frac{1}{a}\sin ax + C$

e.g. $\int \cos(7x)\, dx = \frac{1}{7}\sin(7x) + C$

$\int \sec^2 x\, dx = \tan x + C$, since $(\tan x)' = \sec^2 x$

$\int \csc^2 x\, dx = -\cot x + C$, since $(\cot x)' = -\csc^2 x$

$\int \sec x \tan x\, dx = \sec x + C$, since $(\sec x)' = \sec x \tan x$

$\int \csc x \cot x\, dx = -\csc x + C$, since $(\csc x)' = -\csc x \cot x$

Trigonometric Values to Memorize

θ	**0**	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

Required Theorem Proofs to Memorize (See pages 152 to 155 and pages 283 to 286 for hints to help understand these proofs.)

Theorem 1: (Differentiable Functions are Continuous)

(a) Prove: If a function *f* is differentiable at point *a*, then it is continuous at *a*.

(b) Is it true that if *f* is continuous at *a* it is also differentiable at *a*? Justify your answer.

Proof:

(a) If *f* is differentiable at *a*, then $f'(a)$ exists where

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

$$\begin{aligned}\lim_{h\to 0} f(a+h) &= \lim_{h\to 0} f(a+h)\\ &= \lim_{h\to 0}\left[f(a+h)-f(a)+f(a)\right]\\ &= \lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h}\cdot h+f(a)\right]\\ &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\cdot\lim_{h\to 0} h+\lim_{h\to 0} f(a)\\ &= f'(a)\cdot 0+f(a)\\ &= f(a)\end{aligned}$$

We have proven $\lim_{h\to 0} f(a+h)=f(a)$ meaning *f* is continuous at *a*.

(b) **FALSE**. A function can be continuous at *a* but not be differentiable at *a*. For example, $f(x)=|x|$ is continuous at 0 but it is not differentiable at 0.

Theorem 2: (The Constant Multiple Rule)

Given that *c* is a constant and *f* is a differentiable function, prove:

$$\left(c\cdot f(x)\right)'=c\cdot f'(x).$$

Proof:

$$\begin{aligned}\left(c\cdot f(x)\right)' &= \lim_{h\to 0}\frac{c\cdot f(x+h)-c\cdot f(x)}{h}\\ &= c\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= c\cdot f'(x)\quad \underline{\text{proven}}\end{aligned}$$

Theorem 3: (The Sum Rule)

Given that *f* and *g* are differentiable functions, prove:

$$\left(f(x)+g(x)\right)'=f'(x)+g'(x).$$

Proof:

$$\begin{aligned}\left(f(x)+g(x)\right)' &= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-\left(f(x)+g(x)\right)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h}\\ &= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right]\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= f'(x)+g'(x)\quad \underline{\text{proven}}\end{aligned}$$

Theorem 4: (The Product Rule)

Given that *f* and *g* are differentiable functions, prove:

$$\left(f(x)g(x)\right)'=f'(x)g(x)+f(x)g'(x).$$

Proof:

$$\begin{aligned}\left(f(x)g(x)\right)' &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-g(x+h)f(x)+g(x+h)f(x)-f(x)g(x)}{h}\\ &= \lim_{h\to 0}\left[\frac{f(x+h)g(x+h)-g(x+h)f(x)}{h}+\frac{g(x+h)f(x)-f(x)g(x)}{h}\right]\\ &= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\cdot g(x+h)+f(x)\cdot\frac{g(x+h)-g(x)}{h}\right]\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\cdot\lim_{h\to 0} g(x+h)+\lim_{h\to 0} f(x)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= f'(x)g(x)+f(x)g'(x)\quad \underline{\text{proven}}\end{aligned}$$

(Note that $\lim_{h\to 0} g(x+h)=g(x)$ since *g* is continuous; differentiable functions are continuous.)

Theorem 5: (The Derivative of sin *x*.)

Prove: $(\sin x)'=\cos x$ (i.e. $\frac{d}{dx}\sin x=\cos x$).

Proof:

$$\begin{aligned}(\sin x)' &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\ &= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}\\ &= \lim_{h\to 0}\frac{\sin x\cos h-\sin x+\cos x\sin h}{h}\\ &= \lim_{h\to 0}\left[\frac{\sin x\cos h-\sin x}{h}+\frac{\cos x\sin h}{h}\right]\\ &= \lim_{h\to 0}\left[\sin x\cdot\frac{\cos h-1}{h}+\cos x\cdot\frac{\sin h}{h}\right]\\ &= \lim_{h\to 0}\sin x\cdot\lim_{h\to 0}\frac{\cos h-1}{h}+\lim_{h\to 0}\cos x\cdot\lim_{h\to 0}\frac{\sin h}{h}\\ &= \sin x\cdot 0+\cos x\cdot 1\\ &= \cos x\quad \underline{\text{proven}}\end{aligned}$$

The Mean-Value Theorem

Given the function f which is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists at least one number c in (a, b) such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Theorem 6:

If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

Proof:

Let x_1 and x_2 be <u>any</u> two numbers in (a, b) such that $x_1 < x_2$.

Clearly $f(x)$ is differentiable and continuous on $[x_1, x_2]$. Therefore, the Mean-Value Theorem applies. There exists at least one number c in $[x_1, x_2]$ such that:

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

But, $f'(c) = 0$ since $f'(x) = 0$ on $[x_1, x_2]$. Therefore:

$$0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \rightarrow f(x_2) - f(x_1) = 0 \rightarrow f(x_2) = f(x_1)$$

The only way that $f(x_2) = f(x_1)$ for any two numbers x_1 and x_2 in (a, b) is if f is constant on (a, b). <u>Proven</u>

Definitions of Increasing and Decreasing Functions:

Let x_1 and x_2 be <u>any</u> two numbers in an interval I such that $x_1 < x_2$.

If $f(x_2) > f(x_1)$ on I, then f is an **increasing** function.

If $f(x_2) < f(x_1)$ on I, then f is a **decreasing** function.

Theorem 7: (Increasing Functions)

(a) Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Prove:
If $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.

(b) Is it true that if f is increasing on $[a, b]$, then $f'(x) > 0$ for all x in (a, b)? Justify your answer.

Proof:

(a) Let x_1 and x_2 be <u>any</u> two numbers in $[a, b]$ such that $x_1 < x_2$.

Clearly $f(x)$ is differentiable and continuous on $[x_1, x_2]$. Therefore, the Mean-Value Theorem applies. There exists at least one number c in $[x_1, x_2]$ such that:

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

But, $f'(c) > 0$ since $f'(x) > 0$ on $[x_1, x_2]$. Therefore:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0 \rightarrow f(x_2) - f(x_1) > 0 \rightarrow f(x_2) > f(x_1)$$

Since $f(x_2) > f(x_1)$, for any $x_1 < x_2$ in $[a, b]$, then f is increasing on $[a, b]$. <u>Proven</u>

(b) No, it is <u>not</u> true that if f is increasing on $[a, b]$, then $f'(x) > 0$ for all x in (a, b). For example, $f(x) = x^3$ is increasing on $(-\infty, \infty)$, but $f'(x) \not> 0$ on $(-\infty, \infty)$, $f'(x) = 3x^2 = 0$ when $x=0$ ($f'(x) > 0$ for all other values of x though). i.e., Even though $f(x) = x^3$ is increasing, $f'(x) = 0$ at $x=0$.

Theorem 8: (Decreasing Functions)

(a) Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Prove:
If $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.

(b) Is it true that if f is decreasing on $[a, b]$, then $f'(x) < 0$ for all x in (a, b)? Justify your answer.

Proof:

(a) Let x_1 and x_2 be <u>any</u> two numbers in $[a, b]$ such that $x_1 < x_2$.

Clearly $f(x)$ is differentiable and continuous on $[x_1, x_2]$. Therefore, the Mean-Value Theorem applies. There exists at least one number c in $[x_1, x_2]$ such that:

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

But, $f'(c) < 0$ since $f'(x) < 0$ on $[x_1, x_2]$. Therefore:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} < 0 \rightarrow f(x_2) - f(x_1) < 0 \rightarrow f(x_2) < f(x_1)$$

Since $f(x_2) < f(x_1)$, for any $x_1 < x_2$ in $[a, b]$, then f is decreasing on $[a, b]$. <u>Proven</u>

(b) No, it is <u>not</u> true that if f is decreasing on $[a, b]$, then $f'(x) < 0$ for all x in (a, b). For example, $f(x) = -x^3$ is decreasing on $(-\infty, \infty)$, but $f'(x) \not< 0$ on $(-\infty, \infty)$, $f'(x) = -3x^2 = 0$ when $x=0$ ($f'(x) < 0$ for all other values of x though). i.e., Even though $f(x) = -x^3$ is decreasing, $f'(x) = 0$ at $x=0$.

Lesson 8: Log and Exponential Derivatives

MEMORIZE THE FOLLOWING:

e is a constant $(e = 2.718...)$ and "ln" is the log with base e:

$$\boxed{\ln 1 = 0} \text{ and } \boxed{\ln e = 1}$$

The 3 Log Laws:

$$\log_a(mn) = \log_a m + \log_a n \quad \rightarrow \quad \ln(mn) = \ln m + \ln n$$

$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n \quad \rightarrow \quad \ln\left(\frac{m}{n}\right) = \ln m - \ln n$$

$$\log_a(m^p) = p \cdot \log_a m \quad \rightarrow \quad \ln(m^p) = p \cdot \ln m$$

Derivatives of Exponential and Logarithmic Functions:

$$(e^u)' = e^u \cdot u'$$

$$(a^u)' = a^u \cdot u' \cdot \ln a$$

$$(\ln u)' = \frac{u'}{u}$$ Note that $(\ln|u|)' = \frac{u'}{u}$ also.

$$(\log_a u)' = \frac{u'}{(u)\ln a}$$

$(var)^{var}$ requires **logarithmic differentiation**

Derivative of an Inverse Function:

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

1. Find $\frac{dy}{dx}$ for the following. You need not simplify your answer.

(a) $y = \left(x^3 - 7x^2 + 10\right)e^{x^5+4}$

(b) $y = 3x^5 \ln\left(x - x^4\right)$

(c) $y = \dfrac{e^{x^2+3}}{4\ln x + x^2}$

(d) $y = \left(e^x + 7\right)^3 \ln\left(\dfrac{5}{x} + x^2\right)$

(e) $y = \ln\left|x^{10} - 3x^3 + e^5\right|$

(f) $y = \dfrac{\sqrt{\ln x} + 2}{x^3 + 2x + 1}$

(g) $y = 5^{3x} \cdot x^5 + \log_3\left(1 + x^2\right)$

(h) $y = x^\pi + \pi^x + e^\pi$

(i) $y = \ln\left(x^2 + 1\right) + \log\left(x^3 + 1\right)$

(j) $y = 2^{x^2-x} \log_2\left(x^3 + 7x\right)$

(k) $y = \left[\tan^5(3x)\right]\ln\left(x^2 + \sin e^x\right)$

(l) $y = \ln^3 \sec(3x)$

(m) $y = e^y + \ln\left(x^2 y^3\right)$

(n) $y = (\cos x)^{1-x}$

(o) $y = (\ln x)^{2x}$

(p) $y = x^3 + x^x$

(q) $x^2 + y^2 = (x + y)^{\sin y}$

2. Use logarithmic differentiation to find $f'(x)$ if $f(x) = \dfrac{\sqrt[3]{6-x}\,(2+x)^3}{\sqrt{5+3x^2}\,\left(7-x^3\right)^{10}}$.

3. Find the equation of the line tangent to the curve $\ln(xy) + 2e^{x-y} = y + 1$ at (1, 1).

4. At what x-value(s) does $y = x^3 \ln(4x)$ have a horizontal tangent line?

5. If $f(x) = x^3 + x^2 + 5x - 2$, find $\left(f^{-1}\right)'(-2)$.

6. If $f(x) = \sin x$ on $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, find $\left(f^{-1}\right)'\left(\dfrac{1}{2}\right)$.

Lesson 9: Curve-Sketching

On scrap paper, set up a **Table-of-Values** where you will enter all relevant values of *x* which show up during your work. Essentially, all **Top Zeroes** will be put on this table. Always put $x=0$ on the table immediately which will give you your *y*-intercept. When you add a number to your table, state what it is (*critical point, local max, inflection point, etc.*) and, in the case of *critical* and *singular* points, sketch what the point looks like. Do not record asymptotes on the table, record them next to the table.

Step 1: Gather information from $f(x)$, the function itself.

- The **Top Zeroes** are the ***x*-intercepts** (put these on your Table-of-Values!); the **Bottom Zeroes** are the **Vertical Asymptotes**.
- Check for **Horizontal Asymptotes** by computing the limits as $x \to \pm\infty$.

Step 2: Gather information from $f'(x)$, the first derivative.

- The **Top Zeroes** are the **Critical Points (CP)** (put these on your Table-of-Values!); the **Bottom Zeroes** are probably just the **Vertical Asymptotes** you found earlier, if not, they are **Singular Points (SP)** (which, so far, have never shown up on a test!)
- Make a **sign diagram** for $f'(x)$. The sign diagram identifies where $f(x)$ is **increasing** and where $f(x)$ is **decreasing**.
- Looking at the sign diagram, observe the shape of your critical points and note the shape on your Table-of-Values. For each critical point note on the Table-of-Values whether it is a local maximum, local minimum, or neither.

Step 3: Gather information from $f''(x)$, the second derivative.

- The **Top Zeroes** are the **Inflection Points (IP)** (put these on your Table-of-Values!); the **Bottom Zeroes** are probably just the **Vertical Asymptotes** you found earlier, if not, they are perhaps inflection points also.
- Make a **sign diagram** for $f''(x)$. The sign diagram identifies where $f(x)$ is **concave up** and where $f(x)$ is **concave down** and should confirm the inflection points.

Step 4: Now we are ready to sketch the curve.

- Scan your Table-of-Values and your asymptotes and mark the relevant *x* and *y* values on your axes.
- Plot all the points and draw in the asymptotes you found.
- Draw through points with nice smooth arcs, but never draw through a point of inflection. Arc diagonally towards a point of inflection then stop, lift your pen from the page and be sure to change your concavity before you continue to draw the curve.
- If there is a Vertical Asymptote, make sure there is at least one point plotted on each side of the asymptote, adding a random *x* value to your Table-of-Values if necessary. Remember, as a curve nears a Vertical Asymptote it must either swoop up towards $+\infty$ or swoop down towards $-\infty$.
- If there is a Horizontal Asymptote, the far left side of the graph must begin *parallel* to the asymptote and the far right side of the graph must end *parallel* to the asymptote. Anything can happen in the middle; a curve can even cross a horizontal asymptote.

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

1. Make a sign diagram for the following expressions:

(a) $\dfrac{\left(x^2-4\right)^2\left(x-3\right)^3}{x^2+1}$ **(b)** $\dfrac{\left(x-1\right)^2\left(2x-5\right)^3\left(x+2\right)^{3/5}}{\left(x+4\right)^{4/3}}$

2. Find the horizontal and vertical asymptotes for the following functions:

(a) $y=\dfrac{x^2+3x-5}{x^2-5x-6}$ **(b)** $y=\dfrac{\left(3x-1\right)^2\left(2x+1\right)}{x\left(x+2\right)\left(x-1\right)}$ **(c)** $y=\dfrac{3x^2+5x-2}{x^2-4}$

3. For the functions below:

(i) Find the domain of the function.
(ii) Discuss the symmetry.
(iii) Find the vertical and horizontal asymptotes
(iv) Find the local extremes and give the intervals where the function is increasing and decreasing.
(v) Find the points of inflection and the intervals where the function is concave up and concave down.
(vi) Sketch the graph of the function, labelling all relevant information.

(a) $y=2x\left(x+4\right)^3$, you are given $y'=8\left(x+1\right)\left(x+4\right)^2$, $y''=24\left(x+2\right)\left(x+4\right)$

(b) $y=\dfrac{3x^2-1}{x^3}$, you are given $y'=\dfrac{3-3x^2}{x^4}$, $y''=\dfrac{6x^2-12}{x^5}$

(c) $f(x)=4x+\dfrac{1}{x-1}$, you are given $f'(x)=\dfrac{(2x-1)(2x-3)}{\left(x-1\right)^2}$, $f''(x)=\dfrac{2}{\left(x-1\right)^3}$

(d) $y=xe^{-x^2/2}$, you are given $y'=\left(1-x^2\right)e^{-x^2/2}$, $y''=x\left(x^2-3\right)e^{-x^2/2}$

(e) $y=\left(x^2-1\right)^{2/3}$, you are given $y'=\dfrac{4x}{3\left(x^2-1\right)^{1/3}}$, $y''=\dfrac{4\left(x^2-3\right)}{9\left(x^2-1\right)^{4/3}}$

4. Given $f(x) = \dfrac{x^5}{(x^2-1)^2}$, $f'(x) = \dfrac{x^6 - 5x^4}{(x^2-1)^3}$ and $f''(x) = \dfrac{4x^5 + 20x^3}{(x^2-1)^4}$.

(a) Find the domain of $f(x)$, all zeroes of f and intervals where $f > 0$ and $f < 0$.

(b) Find all vertical and horizontal asymptotes.

(c) Discuss and verify any symmetries of $f(x)$.

(d) Find intervals where f is increasing, where f is decreasing, and all local extrema.

(e) Find intervals where f is concave up, where f is concave down, and all inflection points.

(f) Sketch the curve of $y = f(x)$, illustrating the above information.

5. For the function $f(x) = x^4 - 8x^2 - 3$, find its absolute maximum and absolute minimum values on the interval [–2, 1] and locate the points where they occur.

Lesson 10: Max/Min Word Problems

A problem of this sort is trying to maximize or minimize some quantity Q. For example, you may be asked to find the greatest *volume* (Q is the *volume*, which you wish to maximize); the least *cost* (Q is the *cost*, which you wish to minimize); and so on.

Step 1: **Draw a diagram and label its dimensions.** Generally, the diagram will require two variables, say x and y. If it needs less, great; if it needs more, you missed a way of using only two variables. Use the diagram to help make your equations.

Step 2: Since there are *two variables* generally, you will need *two equations*. I call them the ***Q* equation** and the **Constraint equation**. It does not matter which equation you come up with first.

- The ***Q* equation** relates x and y to Q, the quantity being maximized or minimized, and so will have the form $Q = f(x, y)$ (*i.e.* Q = a formula with x and y in it). This formula is usually obvious, like a known volume or area formula.

- The **Constraint equation** relates x and y to a *constant* that has been given in the problem, and so will have the form $\# = g(x, y)$ (*i.e.* # = a formula with x and y in it). This formula enables you to isolate y (or x, if that is easier) and substitute the result into the Q equation.

In general, we isolate *y* (or *x*) in the Constraint equation and sub that into the Q equation.

Step 3: Once you have substituted, you have $Q = f(x)$, a function of only one variable. Simplify the function in preparation for doing a derivative. Get rid of brackets and pull denominators up as negative exponents so that you can avoid using the Product and Quotient Rules as much as possible.

Step 4: Compute Q' and simplify it. Pull any negative exponents back down to the denominator, get a common denominator if necessary, and factor completely.

Step 5: Do a complete first derivative analysis as in curve-sketching.

- Find *Top Zeroes* (critical points) and *Bottom Zeroes* (assume these are vertical asymptotes).

- Make a sign diagram for Q'. If you are looking for a max, you can bet the local max is your answer; if you are looking for a min, you can bet the local min is your answer.

- If the sign diagram has more than 2 arrows on it, you have not proved the critical point is the answer you are looking for. In these cases, cut the sign diagram down by pointing out x's endpoints. Usually, you can declare $x > 0$, since x is a dimension.

In general, we declare, "*Q* is max (or min) at *x* = the critical point."

Step 6: Now reread the question and make sure you actually answer it!

- Typically, you will need to know both x and y (sub the x-value you found in Step 5 into the *Constraint* equation to get the y-value). You can then use those values to give them what they want.

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

1. The cost of producing q pieces of jewellery is $100+200q-20q^2+q^3/3$ dollars and each piece of jewellery will sell for \$700. How many pieces should be sold to maximize the profit? Justify your answer.

2. Assume that the demand function for a certain product is $q=500e^{-0.02p}$ where p is the price in dollars and q is the number of items sold.

(a) Find the revenue function $R(p)$ as a function of price.

(b) Find the rate of change of the revenue when the price is \$10. Should the price be increased?

(c) What price will maximize revenue? Justify your answer.

3. A rectangular enclosure is to be constructed having one side along an existing long wall and the other three sides fenced. If 100 metres of fence are available, what is the largest possible area for the enclosure?

4. A billboard is to contain 30 square metres of printed area with margins of 2 metres at top and bottom and 1 metre on each side. What outside dimensions will minimize the total area of the billboard?

5. A box is to be made from an 8 foot by 3 foot rectangular sheet of tin by cutting equal squares out of the four corners and bending up the resulting four flaps to make the sides of the box. (The box has no top.) What is the largest possible volume of the box?

6. A rectangular box with an open top and a square base is to have a volume of 320 cubic inches. If the material for the base costs 50 cents per square inch and the material for the four sides costs 40 cents per square inch, find the dimensions of the box that minimize the cost of the material from which it is made.

7. A rectangle has one vertex at the origin, one vertex on the positive x-axis, one vertex on the positive y-axis, and one vertex on the curve $f(x)=e^{-x}$. Of all such rectangles, find the dimensions of the one having the largest area.

8. A cylindrical tin can is to have a volume of 100π cubic centimetres. The top of the can costs twice as much per square centimetre than the bottom or wall. What radius will cost the least? [Hint: the volume of a cylinder is $\pi r^2 h$ while the *lateral* surface area is $2\pi rh$.]

9. Find the point on the graph of $x - y^{3/2} = 1$ that is closest to the point (1, 4).

10. Find the largest possible area for an isosceles triangle if it has a perimeter of 2 metres.

11. A submarine is travelling due east at 30 km/hour and heading straight for a point *P*. A battleship is travelling due south at 20 km/hour and heading for the same point *P*. At 4:00 am, their distances from *P* are 210 km for the submarine and 140 km for the battleship. At what time will they be closest to each other?

12. A lighthouse *L* is located on a small island 5 km north of a point *A* on a straight east-west shoreline. A cable is to be laid from *L* to point *B* on the shoreline 10 km east of *A*. The cable will be laid through the water in a straight line from *L* to a point *C* on the shoreline between *A* and *B* and from there to *B* along the shoreline. If the part of the cable lying in the water costs $5000/km and the part along the shoreline costs $3000/km, where should *C* be chosen to minimize the total cost of the cable?

LESSON 11: Antiderivatives (Integrals)

The Fundamental Theorem of Calculus

Part 1: $\left(\int_a^u f(t)\,dt\right)' = f(u)\cdot u'$

Part 2: $\int_a^b f'(x)dx = f(x)\Big|_a^b = f(b) - f(a)$

(i.e. Fundamental Theorem tells us that integrals are antiderivatives.)

Antiderivative Formulas

$\int K\,dx = Kx + C$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$

$\int \frac{1}{x}\,dx = \ln|x| + C$

$\int e^x\,dx = e^x + C$, so $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$

e.g. $\int e^{5x}\,dx = \frac{1}{5}e^{5x} + C$

$\int a^x\,dx = \frac{a^x}{\ln a} + C$

$\int \sin x\,dx = -\cos x + C$, so $\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$

e.g. $\int \sin(3x)\,dx = -\frac{1}{3}\cos(3x) + C$

$\int \cos x\,dx = \sin x + C$, so $\int \cos ax\,dx = \frac{1}{a}\sin ax + C$

e.g. $\int \cos(7x)\,dx = \frac{1}{7}\sin(7x) + C$

$\int \sec^2 x\,dx = \tan x + C$, since $(\tan x)' = \sec^2 x$

$\int \csc^2 x\,dx = -\cot x + C$, since $(\cot x)' = -\csc^2 x$

$\int \sec x \tan x\,dx = \sec x + C$, since $(\sec x)' = \sec x \tan x$

$\int \csc x \cot x\,dx = -\csc x + C$, since $(\csc x)' = -\csc x \cot x$

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

1. Solve the following definite and indefinite integrals:

(a) $\int\left(6x^5+12x^2-8\right)dx$

(b) $\int\left(e^{3x}+\dfrac{2}{\sqrt[5]{x}}\right)dx$

(c) $\int\left(\dfrac{1}{x}+\dfrac{3}{2x^2}+e^{2x}-\sqrt[3]{x^2}\right)dx$

(d) $\int\left(2+\dfrac{5}{x}\right)dx$

(e) $\int\left(x^2+1\right)\sqrt{x}\,dx$

(f) $\int\dfrac{e^{-5x}+x^5}{3}\,dx$

(g) $\int_0^1\left(\sqrt{x}-8x^3\right)dx$

(h) $\int_1^2(2x-1)^2\,dx$

(i) $\int_0^{\pi/4}(\sin x+\sec 4x\tan 4x)\,dx$

(j) $\int\left(\dfrac{e^{-5x}+\csc^2 x}{3}\right)dx$

(k) $\int_0^4 f(x)dx$ where $f(x)=\begin{cases}3x^2 & \text{if} \quad x<1\\ x^3+2 & \text{if} \quad x\ge 1\end{cases}$

(l) $\int_{-1}^3\left|x^2-4\right|dx$

2. Find $f(x)$ for the following:

(a) $f'(x)=e-\dfrac{1}{2x}, \quad f(1)=e$

(b) $f'(x)=\dfrac{x^2+x+1}{x}, \quad f(-1)=2$

(c) $f''(x)=(3x-1)^2, \quad f(2)=0, \quad f'(1)=0$

3. If the graph of $y=f(x)$ passes through the point (1, –2), and the slope of its tangent line at any point $(x, f(x))$ is given by the equation $12x^2-24x+1$, find $f(x)$.

4. You know that $\int_0^5 f(x)dx = 25$ and $\int_0^2 f(x)dx = 5$, what is $\int_2^5 f(x)dx$.

5. Find the area bounded by the curve $y = 4x^3 + 4x$, the x-axis, and the vertical lines $x=0$ and $x=2$.

6. Find the area of the region of the plane lying below the parabola $y = 1 - x^2$ and above the x-axis.

7. Find the area of the region bounded by the curve $y = x^2 - 4x - 5$, the x-axis, and the lines $x = 3$ and $x = 6$. (Hint: draw a picture.).

8. Find the area bounded between the x-axis and the curve $y = \cos x - \sin x$ over the interval $0 \le x \le \pi$.

9. The acceleration a at any time t of a particle is given by the formula $\cos t + \sin t$. Its initial position was 0 and its initial velocity was 5. What is the velocity v and position s of the particle at time t.

10. Find $f'(x)$ for the following functions:

(a) $f(x) = \int_1^x \frac{dt}{1+t^3}$

(b) $f(x) = \int_{2x}^5 \sqrt[3]{\cos t}\ dt$

(c) $f(x) = \sin x \int_2^{x^2} e^{t^2}\ dt$

(d) $f(x) = \int_{\ln(x)}^{x^3} \frac{y^3+5}{y^7+9}\ dy$

11. Let f be a function. If $x^2 \ln(x) = \int_0^x f(t)\ dt$ for each $x > 0$, find the value of $f'(1)$.

12. If $F(x) = \int_3^x (t-3)e^t\ dt$, find the point where the graph of $y = F(x)$ has a horizontal tangent line.

13. Note: I am quite certain that you will never have a question like this on your exam although, technically, it is part of your outline. Once in over twenty years they had a question like this on a final, and then they actually ended up calling it a bonus question because so few students knew how to do it. (Many profs had not even taught the students how to do it.) I really include this question for the benefit of those of you taking the course by correspondence since there is inevitably a similar question on the hand-in assignments. (Again, I am confident you would not have such a question on your exam either.)

Compute the Riemann sum for the following functions over the given interval using n equal partitions. Then determine the limit as n approaches infinity ($n \to \infty$). Which is to say, find the area between the given function and the x-axis over the given region using the *definition* of the definite integral. Check your answer by computing the associated definite integral. You will need to know the following formulas:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{b-a}{n}\cdot f\left(a+\frac{(b-a)i}{n}\right)$$

$$\sum_{i=1}^{n}1 = n \qquad \sum_{i=1}^{n}i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n}i^2 = \frac{n(n+1)(2n+1)}{6}$$

(a) $f(x) = 3x^2 - 4$ on [0, 4]

(b) $f(x) = x^2 - 4x + 3$ on [–1, 7]