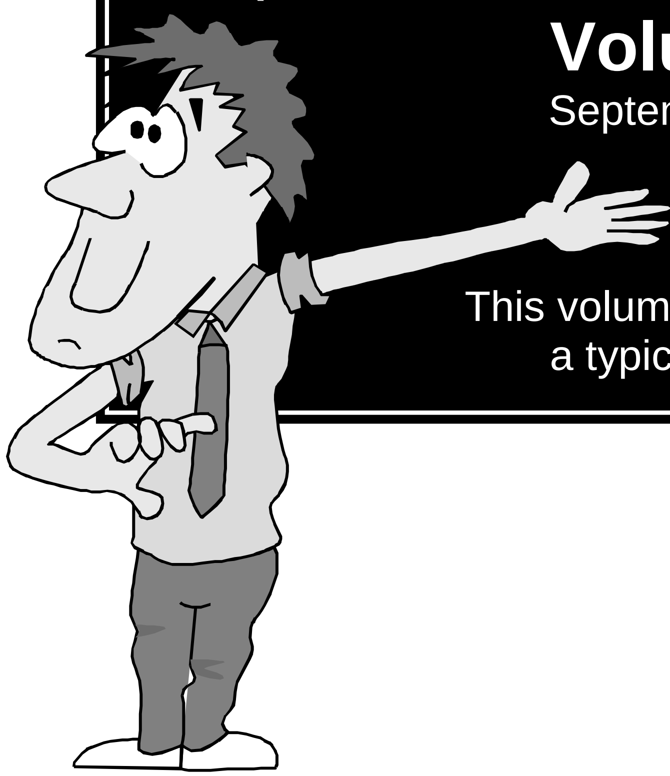
Grant's Tutoring

INTRO CALCULUS

(DIFFERENTIATION and APPLICATIONS)

Volume 1 of 2

September 2011 edition



This volume covers the topics on a typical midterm exam.

Learn What You Need to Know

Know What You Need to Learn

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HOW TO USE THIS BOOK

I have broken the course up into lessons. Study each lesson until you can do all of my lecture problems from start to finish without any help. Then do the Practise Problems for that lesson. If you are able to solve all the Practise Problems I have given you, then you should have nothing to fear about your Midterm or Final Exam.

I have presented the course in what I consider to be the most logical order. Although my books are designed to follow the course syllabus, it is possible your prof will teach the course in a different order or omit a topic. It is also possible he/she will introduce a topic I do not cover. **Make sure you are attending your class regularly! Stay current with the material, and be aware of what topics are on your exam. Never forget, it is your prof that decides what will be on the exam, so pay attention.**

If you have any questions or difficulties while studying this book, or if you believe you have found a mistake, do not hesitate to contact me. My phone number and website are noted at the bottom of every page in this book. "Grant's Tutoring" is also in the phone book. **I welcome your input and questions.**

Wishing you much success,

Grant Skene

Owner of Grant's Tutoring

Formulas and Definitions to Memorize

The Definition of Continuity: $f(x)$ is continuous at $x=a$ if and only if $\lim_{x \to a} f(x) = f(a)$.

The Definition of Derivative: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

The Power Rule: $(x^n)' = nx^{n-1}$

The Product Rule: $(f \cdot g)' = f'g + fg'$

The Quotient Rule: $\left(\dfrac{T}{B}\right)' = \dfrac{T'B - TB'}{B^2}$

The Chain Rule: $(f(u))' = f'(u) \cdot u'$

The Chain Rule Version of Power Rule: $(u^n)' = nu^{n-1} \cdot u'$

Derivatives of Trigonometric Functions:

$(\sin u)' = \cos u \cdot u'$ $\quad (\tan u)' = \sec^2 u \cdot u'$ $\quad (\sec u)' = \sec u \tan u \cdot u'$

$(\cos u)' = -\sin u \cdot u'$ $\quad (\cot u)' = -\csc^2 u \cdot u'$ $\quad (\csc u)' = -\csc u \cot u \cdot u'$

Derivatives of Exponential and Logarithmic Functions:

$(e^u)' = e^u \cdot u'$ $\quad (\ln u)' = \dfrac{u'}{u}$

$(a^u)' = a^u \cdot u' \cdot \ln a$ $\quad (\log_a u)' = \dfrac{u'}{(u)\ln a}$

Derivative of an Inverse Function: $(f^{-1})'(a) = \dfrac{1}{f'(f^{-1}(a))}$

Fundamental Theorem of Calculus: $\left(\int_a^u f(t)\,dt\right)' = f(u) \cdot u'$

Antiderivative Formulas:

$\int K\,dx = Kx + C$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$

$\int \frac{1}{x}\,dx = \ln|x| + C$

$\int e^x\,dx = e^x + C$, so $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$

e.g. $\int e^{5x}\,dx = \frac{1}{5}e^{5x} + C$

$\int a^x\,dx = \frac{a^x}{\ln a} + C$

$\int \sin x\,dx = -\cos x + C$, so $\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$

e.g. $\int \sin(3x)\,dx = -\frac{1}{3}\cos(3x) + C$

$\int \cos x\,dx = \sin x + C$, so $\int \cos ax\,dx = \frac{1}{a}\sin ax + C$

e.g. $\int \cos(7x)\,dx = \frac{1}{7}\sin(7x) + C$

$\int \sec^2 x\,dx = \tan x + C$, since $(\tan x)' = \sec^2 x$

$\int \csc^2 x\,dx = -\cot x + C$, since $(\cot x)' = -\csc^2 x$

$\int \sec x \tan x\,dx = \sec x + C$, since $(\sec x)' = \sec x \tan x$

$\int \csc x \cot x\,dx = -\csc x + C$, since $(\csc x)' = -\csc x \cot x$

Trigonometric Values to Memorize

θ	**0**	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

Required Theorem Proofs to Memorize
(See pages 152 to 155 for hints to help understand these proofs.)

Theorem 1: (Differentiable Functions are Continuous)

(a) Prove: If a function f is differentiable at point a, then it is continuous at a.

(b) Is it true that if f is continuous at a it is also differentiable at a? Justify your answer.

Proof:

(a) If f is differentiable at a, then $f'(a)$ exists where

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

$$\begin{aligned}
\lim_{h \to 0} f(a+h) &= \lim_{h \to 0} f(a+h) \\
&= \lim_{h \to 0} \left[f(a+h) - f(a) + f(a) \right] \\
&= \lim_{h \to 0} \left[\frac{f(a+h) - f(a)}{h} \cdot h + f(a) \right] \\
&= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \cdot \lim_{h \to 0} h + \lim_{h \to 0} f(a) \\
&= f'(a) \cdot 0 + f(a) \\
&= f(a)
\end{aligned}$$

We have proven $\lim_{h \to 0} f(a+h) = f(a)$ meaning f is continuous at a.

(b) **FALSE**. A function can be continuous at a but not be differentiable at a. For example, $f(x) = |x|$ is continuous at 0 but it is not differentiable at 0.

Theorem 2: (The Constant Multiple Rule)

Given that c is a constant and f is a differentiable function, prove:

$$\left(c \cdot f(x)\right)' = c \cdot f'(x).$$

Proof:

$$\begin{aligned}
\left(c \cdot f(x)\right)' &= \lim_{h \to 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h} \\
&= c \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= c \cdot f'(x) \quad \text{proven}
\end{aligned}$$

Theorem 3: (The Sum Rule)

Given that f and g are differentiable functions, prove:

$$\left(f(x) + g(x)\right)' = f'(x) + g'(x).$$

Proof:

$$\begin{aligned}
\left(f(x) + g(x)\right)' &= \lim_{h \to 0} \frac{f(x+h) + g(x+h) - \left(f(x) + g(x)\right)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h} \\
&= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right] \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x) + g'(x) \quad \text{proven}
\end{aligned}$$

Theorem 4: (The Product Rule)

Given that f and g are differentiable functions, prove:

$$\left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x).$$

Proof:

$$\begin{aligned}
\left(f(x)g(x)\right)' &= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h)g(x+h) - g(x+h)f(x) + g(x+h)f(x) - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \left[\frac{f(x+h)g(x+h) - g(x+h)f(x)}{h} + \frac{g(x+h)f(x) - f(x)g(x)}{h} \right] \\
&= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \cdot g(x+h) + f(x) \cdot \frac{g(x+h) - g(x)}{h} \right] \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h \to 0} g(x+h) + \lim_{h \to 0} f(x) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x)g(x) + f(x)g'(x) \quad \text{proven}
\end{aligned}$$

(Note that $\lim_{h \to 0} g(x+h) = g(x)$ since g is continuous; differentiable functions are continuous.)

Theorem 5: (The Derivative of sin *x*.)

Prove: $(\sin x)' = \cos x$ (i.e. $\frac{d}{dx} \sin x = \cos x$).

Proof:

$$\begin{aligned}
(\sin x)' &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\sin x \cos h - \sin x + \cos x \sin h}{h} \\
&= \lim_{h \to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right] \\
&= \lim_{h \to 0} \left[\sin x \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h} \right] \\
&= \lim_{h \to 0} \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \lim_{h \to 0} \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h} \\
&= \sin x \cdot 0 + \cos x \cdot 1 \\
&= \cos x \quad \text{proven}
\end{aligned}$$

Lesson 2: Limits

Memorize these two trig limits: $\lim_{h \to 0} \frac{\sin h}{h} = 1$ and $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

For each of **questions 1 to 14**, find the value of the limit, if it exists. If it does not exist, is it infinity, negative infinity, or neither? Justify your answers.

1. $\lim_{x \to -1} \frac{x^2 - x - 2}{x + 1}$

2. $\lim_{x \to -2} \frac{x + 2}{\sqrt{x+4} - \sqrt{2}}$

3. $\lim_{x \to 3} \frac{4 - \sqrt{x^2 + 7}}{4x^2 - 5x - 21}$

4. $\lim_{x \to 2} \frac{|x - 2|}{x - 2}$

5. $\lim_{x \to -1^+} \frac{x^2 - 3x - 4}{|5x^2 + 9x + 4|}$

6. $\lim_{x \to 5} \frac{3 - x}{x^2 - 10x + 25}$

7. $\lim_{x \to 2^-} \frac{x^2 - 9}{x^2 - x - 2}$

8. $\lim_{x \to 6} \sqrt{x - 6}$

9. $\lim_{x \to -2^-} \sqrt{x^2 - 7x - 18}$

10. $\lim_{x \to \infty} \frac{2x^3 - x - 2}{1 + 3x^3}$

11. $\lim_{x \to -\infty} \frac{4x^3 + 3x - 2}{3x^2 + 5x + 1}$

12. $\lim_{x \to -\infty} \frac{5x - 21}{\sqrt{4x^2 - 3x + 7}}$

13. $\lim_{x \to \infty} \frac{(2x-3)^2 (x^2 + 6x + 5)}{x^4 + x^3 + 6}$

14. $\lim_{x \to -\infty} \frac{(1 - 3x)\sqrt{4x^2 + 5}}{(5x + 4)^2}$

15. Which limits in questions 1 to 14 above indicate the existence of a Vertical or Horizontal Asymptote?

16. Find the following limits.

(a) $\lim_{x \to 0} \frac{\sin(2x) + 7x}{3x}$ **(b)** $\lim_{x \to 0} \frac{\sin(2x)}{\sin(5x)}$ **(c)** $\lim_{x \to 0} \frac{\tan(4x)}{2x}$

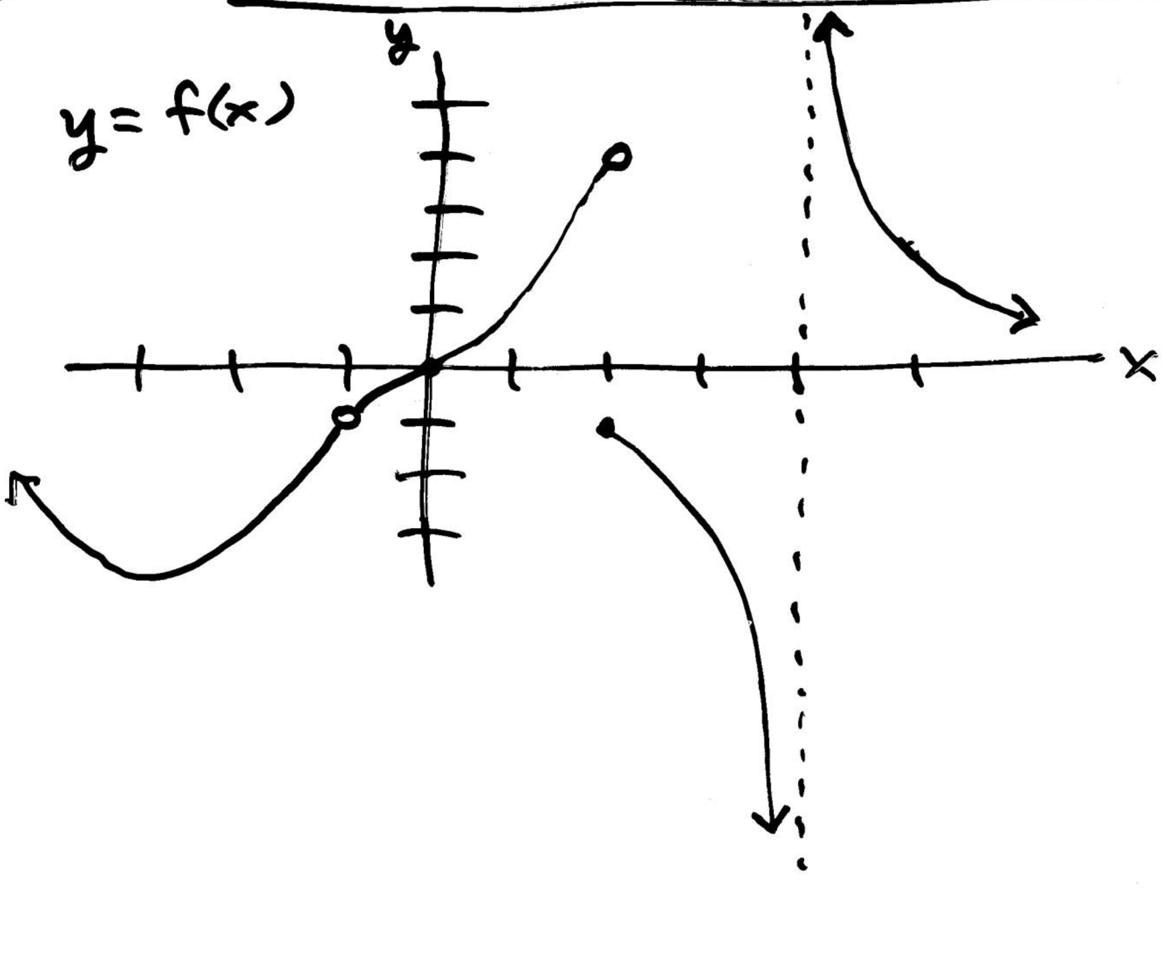
17. Use the Squeeze Theorem to solve the following limits.

(a) $\lim_{x \to 0} \sqrt{x^4 \sin^2\left(\frac{1}{x}\right)}$ **(b)** $\lim_{x \to 0} x^5 \cos\left(\frac{3}{x^3}\right)$

LIMITS

When asked to find the Limit of a function as x approaches a given value, we are really trying to find the y-value, f(x)'s graph is tending towards as x approaches its value.

Key: **A Limit is a y-value.**



For the graph above, answer the following questions.

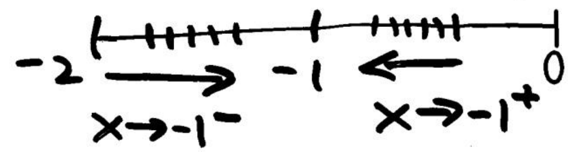
1. $f(-1)$ → reads f "at" -1
   ie what is the y-value <u>at</u> $x=-1$
   $f(-1)$ is undefined (b/c there is no dot when $x=-1$ and so no y-value when $x=-1$.)

2. $f(1)=1$ (b/c dot at $(1,1)$ ie $y=1$ when $x=1$.)

3. $f(2)=-1$ (b/c $y=-1$ at $x=2$ dot at $(2,-1)$)

4. $f(4)$ is undefined

5. $\lim\limits_{x\to -1^-} f(x)$ → reads find the Limit of $f(x)$ as x approaches -1 from the left side of -1



ie Follow the graph of $f(x)$ as we approach $x=-1$ from the Left side and see what y-value the graph approaches. The y-value we identify is the LIMIT.

$\lim_{x \to -1^-} f(x) = -1$ b/c the graph leads to the y-value of -1 as $x \to -1^-$

6. $\lim_{x \to -1^+} f(x) = -1$ (b/c the graph leads to y-value of -1 as $x \to -1^+$)

7. $\lim_{x \to -1} f(x)$

<u>Note</u>: $x \to -1$ implies $\begin{cases} x \to -1^- \\ x \to -1^+ \end{cases}$

ie that we have approached -1 from both sides.

$\lim_{x \to -1} f(x)$ exists if and only if

$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x)$

ie (LHL = RHL)

<u>Then</u>: $\boxed{\lim_{x \to -1} f(x) = -1}$

b/c LHL $= -1$ <u>and</u> RHL $= -1$

8. $\lim_{x \to 1^-} f(x) = 1$ (b/c y-value $= 1$ as $x \to 1^-$)

9. $\lim_{x \to 1^+} f(x) = 1$ also

10. $\lim_{x \to 1} f(x) = 1$ b/c LHL = 1 <u>and</u> RHL = 1

11. $\lim_{x \to 2^-} f(x) = 4$ (led to y-value of 4 as $x \to 2^-$)

12. $\lim_{x \to 2^+} f(x) = -1$ (led to y-value of −1 as $x \to 2^+$)

13. $\lim_{x \to 2} f(x)$ does not exist
b/c LHL = 4, RHL = −1
LHL ≠ RHL

14. $\lim_{x \to 4^-} f(x) = -\infty$
b/c y-value is becoming infinitely negative as $x \to 4^-$

15. $\lim_{x \to 4^+} f(x) = \infty$
b/c y-value is becoming infinitely positive as $x \to 4^+$

16. $\lim_{x \to 4} f(x)$ does not exist
b/c LHL = $-\infty$, RHL = ∞
LHL ≠ RHL

Lesson 3: Continuity

Memorize the Definition of Continuity:

$f(x)$ is continuous at $x=a$ if and only if $\lim_{x \to a} f(x) = f(a)$.

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

1. For the function shown below, determine for which x it is continuous. Justify your answer.

$$f(x) = \begin{cases} x-2 & \text{if} \quad x \le -2 \\ \dfrac{x^2-4}{x+2} & \text{if} \quad -2 < x < 1 \\ 4-x & \text{if} \quad 1 \le x \end{cases}$$

2. Is $g(x)$ below continuous at $x=1$? (Justify your answer.)

$$g(x) = \begin{cases} \dfrac{x-1}{\sqrt{x^2+3}-2} & x \ne 1 \\ 3 & x = 1 \end{cases}$$

3. For $f(x)$ below, find a and b which will make the function continuous everywhere.

$$f(x) = \begin{cases} x^2+4 & \text{if} \quad x \le 0 \\ ax+b & \text{if} \quad 0 < x < 3 \\ 6/x & \text{if} \quad x \ge 3 \end{cases}$$

4. Show that $f(x) = x^3 + 3x - 1$ has a zero between $x=0$ and $x=1$.

5. Show that $f(x) = x^3 - x^2 - 7x - 4$ has at least three zeros on $[-4, 4]$.

Lesson 4: The Definition of Derivative

Memorize the Definition of Derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

1. For $f(x) = 2x^2 + 3x + 1$:

(a) Find the average rate of change of $f(x)$ for the interval $0 \le x \le 2$.

(b) Use the definition of derivative to find the instantaneous rate of change of $f(x)$.

(c) Find the equation of the tangent line to $y = f(x)$ at $x=1$ in $y = mx + b$ form.

2. For the functions below, find $f'(x)$ using only the definition of derivative.

(a) $f(x) = \dfrac{2x}{9 - x^2}$

(b) $f(x) = \dfrac{1}{\sqrt{x+1}}$

LESSON 5: Differentiation Rules

All of the formulas on this page must be MEMORIZED

The Power Rule: $\left(x^n\right)' = nx^{n-1}$

The Product Rule: $\left(f \cdot g\right)' = f'\,g + f\,g'$

The Quotient Rule: $\left(\dfrac{T}{B}\right)' = \dfrac{T'\,B - T\,B'}{B^2}$

The Chain Rule: $\left(f\left(u\right)\right)' = f'\left(u\right) \cdot u'$

The Chain Rule Version of Power Rule: $\left(u^n\right)' = nu^{n-1} \cdot u'$

Derivatives of Trigonometric Functions:

$\left(\sin u\right)' = \cos u \cdot u'$ $\quad\left(\tan u\right)' = \sec^2 u \cdot u'$ $\quad\left(\sec u\right)' = \sec u \tan u \cdot u'$

$\left(\cos u\right)' = -\sin u \cdot u'$ $\quad\left(\cot u\right)' = -\csc^2 u \cdot u'$ $\quad\left(\csc u\right)' = -\csc u \cot u \cdot u'$

Derivatives of Exponential and Logarithmic Functions:

$\left(e^u\right)' = e^u \cdot u'$ $\quad\left(\ln u\right)' = \dfrac{u'}{u}$

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

1. Find the indicated derivatives for the following. You need not simplify your answer.

(a) $y = \sqrt[3]{x^2} + \dfrac{2}{x^3} + \dfrac{4x}{5}$, find $\dfrac{dy}{dx}$

(b) $y = \sqrt{2} - 5x^2 + \dfrac{12}{5x^7}$, find y'

(c) $y = \dfrac{x + 9x^5 + 1}{3x + 2}$, find y'

(d) $y = \left(x^2 + 1\right)^{10}$, find $\dfrac{dy}{dx}$

(e) $y = \left(4 - 6x^2\right)^8$, find y'

(f) $y = 2x^7\sqrt{x^2 + 1}$, find y'

(g) $y = e^{2x+1}$, find y'

(h) $f(t) = e^{3t^2} + 4t^3$, find $f'(1)$

(i) $f(x) = \sqrt[3]{x^4 + \left(x + x^3\right)^{-2}}$, find $f'(x)$

(j) $y = (x+1)\left(x^2+1\right)^3\left(x^3-1\right)^4$, find y'

(k) $y = \dfrac{x^3\sqrt[5]{x+2}}{x^3 + 2x + 1}$, find y'

(l) $y = \left(x^2 + \pi^2\right)\left(\sqrt[5]{\left(x^2 + x + 3\right)^4}\right)$, find y'

(m) $y = \ln\left(x^{10} - 3x^3 + e^5\right)$, find $\dfrac{dy}{dx}$

(n) $y = \dfrac{\sqrt{\ln x} + 2}{e^x + 3}$, find $\dfrac{dy}{dx}$

(o) $Q = 5\sin(2\pi t + 3)$, find $\dfrac{dQ}{dt}$

(p) $C = \sin^5 x$, find $\dfrac{dC}{dx}$

(q) $y = \tan x^7 \cos^2 x$, find y'

(r) $y = \left[\sec\left(1 + 4x^5\right)\right]^{3/2}$, find y'

(s) $y = \sqrt{x^3 + x}\,\csc(1/x)$, find $\dfrac{dy}{dx}$

(t) $y = \dfrac{\cot(5x)}{\tan^5(2x) + x}$, find $\dfrac{dy}{dx}$

2. Find an equation of the tangent line to the curve $y = \left(x^3 + x^2 + 2\right)^5$ when $x = -2$.

3. Find the equation of the normal line to the curve whose equation is $y = 2 + \sqrt{x^2 + 3}$ at the point on the curve with x-coordinate 1.

4. Given $f(x) = (2x + 5)^3$, compute $f''(-2)$.

5. A point moves along the x-axis such that its position at any time t is given by the function $x = 5t^2 - 3t^3$. Find the position, velocity and acceleration of the point when t=2.

6. We are given $y = f\left(x^2 + x\right)$, where f is an unknown differentiable function.

We know $f(2) = -5$ and $f'(2) = 6$. Compute $\left.\dfrac{dy}{dx}\right|_{x=1}$.

Be sure to MEMORIZE the PROOFS of THEOREMS 1 to 5.

One of these proofs is 99% certain to be on your U of M Midterm Exam.

Note: Anything I put in squiggly brackets "{ }" during the proofs are merely hints to help you understand how to construct the proof. Do not write these hints down while you are doing the proof on an exam. See **page 2** above for the exact way to write the proof.

Required Theorem 1: (Differentiable Functions are Continuous)

(a) Prove: If a function f is differentiable at point a, then it is continuous at a.

(b) Is it true that if f is continuous at a it is also differentiable at a? Justify your answer.

Proof:

{This proof has been asked very frequently on exams.}

(a) If f is differentiable at a, then $f'(a)$ exists where $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$.

{We are simply stating the Definition of Derivative at a. We need to prove it follows that $\lim_{h \to 0} f(a+h) = f(a)$, an alternate form of the Definition of Continuity.}

$$
\begin{aligned}
\lim_{h \to 0} f(a+h) &= \lim_{h \to 0} f(a+h) \\
&= \lim_{h \to 0} \left[f(a+h) - f(a) + f(a) \right] \\
&= \lim_{h \to 0} \left[\frac{f(a+h)-f(a)}{h} \cdot h + f(a) \right] \\
&= \underbrace{\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}}_{\{=f'(a),\text{ by definition}\}} \cdot \underbrace{\lim_{h \to 0} h}_{\{=0,\text{ simply subbing in } h=0\}} + \underbrace{\lim_{h \to 0} f(a)}_{\left\{\substack{=0,\text{ since there is} \\ \text{no } h \text{ to replace with } h=0}\right\}} \\
&= f'(a) \cdot 0 + f(a) \\
&= f(a)
\end{aligned}
$$

We have proven $\lim_{h \to 0} f(a+h) = f(a)$ meaning f is continuous at a.

(b) **FALSE**. A function can be continuous at a but not be differentiable at a. For example, $f(x) = |x|$ is continuous at 0 but it is not differentiable at 0. {This example is sufficient to prove this statement false. $f(x) = |x|$ is a "V"-shaped graph with the point of the V at 0. You can't draw a tangent line at a sharp point. No tangent line means no derivative → not differentiable.}

Required Theorem 2: (The Constant Multiple Rule)

Given that c is a constant and f is a differentiable function, prove: $\left(c \cdot f(x)\right)' = c \cdot f'(x)$.

Proof:

{This proof is rarely on exams; probably because it is too easy. Make sure you know it anyway!}

{We are proving what I call the Coefficient Rule. This is simply a matter of applying the Definition of Derivative and factoring out the c.}

$$\left(c \cdot f(x)\right)' = \lim_{h \to 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h} = c \cdot \underbrace{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}_{= f'(x), \text{ by definition}} = c \cdot f'(x) \ \underline{\text{proven}}$$

Required Theorem 3: (The Sum Rule)

Given that f and g are differentiable functions, prove: $\left(f(x) + g(x)\right)' = f'(x) + g'(x)$.

Proof:

{This proof has been asked frequently on exams. It is simply a matter of applying the Definition of Derivative and separating the "f" and "g" parts.}

$$\begin{aligned}
\left(f(x) + g(x)\right)' &= \lim_{h \to 0} \frac{f(x+h) + g(x+h) - \left(f(x) + g(x)\right)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h} \\
&= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}\right] \\
&= \underbrace{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}_{\{= f'(x), \text{ by definition}\}} + \underbrace{\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}}_{\{= g'(x), \text{ by definition}\}} \\
&= f'(x) + g'(x) \quad \underline{\text{proven}}
\end{aligned}$$

Required Theorem 4: (The Product Rule)

Given that f and g are differentiable functions, prove: $\left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x)$.

Proof:

{This proof has been asked very frequently on exams; probably because it is pretty involved. Again, we are applying the Definition of Derivative but we have to do some tricky algebra to get everything to work.}

$$\left(f(x)g(x)\right)' = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

{We are going to take the second part of the first term "$g(x+h)$" and the first part of the second term "$f(x)$", and multiply them together: "$g(x+h)f(x)$". We will then insert this into the numerator in order to set up some factoring.}

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) \overbrace{- g(x+h)f(x) + g(x+h)f(x)}^{\left\{\substack{\text{We have subtracted } g(x+h)f(x) \text{ and then} \\ \text{added it right back, so technically nothing has changed.}}\right\}} - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\left[\underbrace{\frac{f(x+h)g(x+h) - g(x+h)f(x)}{h}}_{\{\text{factor } g(x+h) \text{ out to the right side}\}} + \underbrace{\frac{g(x+h)f(x) - f(x)g(x)}{h}}_{\{\text{factor } f(x) \text{ out to the left side}\}} \right]$$

$$= \lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h} \cdot g(x+h) + f(x) \cdot \frac{g(x+h) - g(x)}{h} \right]$$

$$= \underbrace{\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}}_{\{= f'(x), \text{ by definition}\}} \cdot \underbrace{\lim_{h\to 0} g(x+h)}_{\{= g(x), \text{ see below}\}} + \underbrace{\lim_{h\to 0} f(x)}_{\{= f(x)\}} \cdot \underbrace{\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}}_{\{= g'(x), \text{ by definition}\}}$$

$= f'(x)g(x) + f(x)g'(x)$ <u>proven</u>

(Note that $\lim_{h \to 0} g(x+h) = g(x)$ since g is continuous; differentiable functions are continuous.) {Make sure you point this out or you will lose marks!}

<u>Required Theorem 5:</u> (The Derivative of sin *x*.)

Prove: $(\sin x)' = \cos x$ (i.e. $\frac{d}{dx}\sin x = \cos x$).

<u>Proof:</u>

{This proof has been asked very frequently on exams; probably because it is pretty involved. Again, we are applying the Definition of Derivative, but we have to use a lot of memorized facts.}

{Make sure you have memorized the following facts:

The two key trig limits: $\lim\limits_{h\to 0}\frac{\sin h}{h} = 1$ and $\lim\limits_{h\to 0}\frac{\cos h - 1}{h} = 0$

The trigonometric identity: $\sin(x+h) = \sin x\cos h + \cos x\sin h$ }

$$(\sin x)' = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$

{We now switch the order of the second and third terms in order to group the two " $\sin x$ " terms together.}

$$= \lim_{h\to 0}\frac{\sin x\cos h - \sin x + \cos x\sin h}{h}$$

$$= \lim_{h\to 0}\left[\underbrace{\frac{\sin x\cos h - \sin x}{h}}_{\{\text{factor } \sin x \text{ out to the left side}\}} + \underbrace{\frac{\cos x\sin h}{h}}_{\{\text{pull } \cos x \text{ out to the left side}\}}\right]$$

$$= \lim_{h\to 0}\left[\sin x\cdot\frac{\cos h - 1}{h} + \cos x\cdot\frac{\sin h}{h}\right]$$

$$= \underbrace{\lim_{h\to 0}\sin x}_{\{=\sin x\}}\cdot\underbrace{\lim_{h\to 0}\frac{\cos h - 1}{h}}_{\{=0\}} + \underbrace{\lim_{h\to 0}\cos x}_{\{=\cos x\}}\cdot\underbrace{\lim_{h\to 0}\frac{\sin h}{h}}_{\{=1\}}$$

$$= \sin x\cdot 0 + \cos x\cdot 1$$

$= \cos x$ <u>proven</u>

Optional Proofs to Memorize

These proofs are unlikely to appear on a U of M exam but can be memorized by the student who wants to be ready for anything.
ONLY MEMORIZE THESE PROOFS IF YOU HAVE LEARNED EVERYTHING ELSE.

Optional Theorem A: (The Quotient Rule)

Given that f and g are differentiable functions, prove: $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)g(x)-f(x)g'(x)}{\left[g(x)\right]^2}$.

Proof:

{This is similar to the Product Rule proof. They have never asked for this proof on a U of M exam and probably never will. We begin by using the Definition of Derivative and simplifying the "Triple Decker" that appears.}

$$\left(\frac{f(x)}{g(x)}\right)' = \lim_{h\to 0}\frac{\dfrac{f(x+h)}{g(x+h)}-\dfrac{f(x)}{g(x)}}{h} = \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x+h)}{h\cdot g(x+h)\cdot g(x)}$$

{We are going to take the second part of the first term "$g(x)$" and the first part of the second term "$f(x)$", and multiply them together: "$g(x)f(x)$". We will then insert this into the numerator, subtracting it first, then adding it back at the end, in order to set up some factoring.}

$$= \lim_{h\to 0}\frac{f(x+h)g(x)\overbrace{-\;g(x)f(x)}\;-\;f(x)g(x+h)\overbrace{+\;g(x)f(x)}}{h\cdot g(x+h)\cdot g(x)}$$

$$= \lim_{h\to 0}\left[\underbrace{\frac{f(x+h)g(x)-g(x)f(x)}{h\cdot g(x+h)\cdot g(x)}}_{\{\text{factor } g(x) \text{ out to the right side}\}} + \underbrace{\frac{-f(x)g(x+h)+g(x)f(x)}{h\cdot g(x+h)\cdot g(x)}}_{\{\text{factor } -f(x) \text{ out to the left side}\}}\right] \quad \{\text{Watch the signs!}\}$$

$$= \lim_{h\to 0}\left[\frac{\left[f(x+h)-f(x)\right]g(x)}{h\cdot g(x+h)\cdot g(x)} + \frac{-f(x)\left[g(x+h)-g(x)\right]}{h\cdot g(x+h)\cdot g(x)}\right]$$

$$= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\cdot\frac{g(x)}{g(x+h)\cdot g(x)} - \frac{f(x)}{g(x+h)\cdot g(x)}\cdot\frac{g(x+h)-g(x)}{h}\right]$$

$$= \underbrace{\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}_{\{=f'(x),\text{ by definition}\}}\cdot\underbrace{\lim_{h\to 0}\frac{g(x)}{g(x+h)\cdot g(x)}}_{\left\{=\frac{g(x)}{g(x)\cdot g(x)}=\frac{g(x)}{[g(x)]^2}\right\}} - \underbrace{\lim_{h\to 0}\frac{f(x)}{g(x+h)\cdot g(x)}}_{\left\{=\frac{f(x)}{g(x)\cdot g(x)}=\frac{f(x)}{[g(x)]^2}\right\}}\cdot\underbrace{\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}}_{\{=g'(x),\text{ by definition}\}}$$

$$= f'(x)\cdot\frac{g(x)}{\left[g(x)\right]^2} - \frac{f(x)}{\left[g(x)\right]^2}\cdot g'(x) = \frac{f'(x)g(x)}{\left[g(x)\right]^2} - \frac{f(x)g'(x)}{\left[g(x)\right]^2} = \frac{f'(x)g(x)-f(x)g'(x)}{\left[g(x)\right]^2} \quad \underline{\text{proven}}$$

(Note that $\lim_{h\to 0} g(x+h) = g(x)$ since g is continuous, differentiable functions are continuous.) {Make sure you point this out or you will lose marks!}

Optional Theorem B: **(The Derivative of cos *x*.)**

Prove: $(\cos x)' = -\sin x$ (i.e. $\frac{d}{dx}\cos x = -\sin x$).

Proof:

{This proof has never been asked on a U of M exam, but I could see the day when it will be asked. It is very similar to the derivative of sin *x* proof: we are applying the Definition of Derivative, but we have to use a lot of memorized facts.}

{Make sure you have memorized the following facts:

The two key trig limits: $\lim_{h \to 0} \frac{\sin h}{h} = 1$ and $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$

The trigonometric identity: $\cos(x+h) = \cos x \cos h - \sin x \sin h$ }

$$(\cos x)' = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

{We now switch the order of the second and third terms in order to group the two "$\cos x$" terms together.}

$$= \lim_{h \to 0} \frac{\cos x \cos h - \cos x - \sin x \sin h}{h}$$

$$= \lim_{h \to 0} \left[\underbrace{\frac{\cos x \cos h - \cos x}{h}}_{\{\text{factor } \cos x \text{ out to the left side}\}} - \underbrace{\frac{\sin x \sin h}{h}}_{\{\text{pull } \sin x \text{ out to the left side}\}} \right]$$

$$= \lim_{h \to 0} \left[\cos x \cdot \frac{\cos h - 1}{h} - \sin x \cdot \frac{\sin h}{h} \right]$$

$$= \underbrace{\lim_{h \to 0} \cos x}_{\{=\cos x\}} \cdot \underbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}}_{\{=0\}} - \underbrace{\lim_{h \to 0} \sin x}_{\{=\sin x\}} \cdot \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{\{=1\}}$$

$$= \cos x \cdot 0 - \sin x \cdot 1$$

$= -\sin x$ <u>proven</u>

Now that the derivatives of sin *x* and cos *x* have been proven, it is pretty straightforward to prove the derivatives of the other four trig functions using the Differentiation Rules. It is merely a matter of using the fact that all trig functions can be written in terms of sine or cosine and applying some trig identities.

Recall: $\tan x = \dfrac{\sin x}{\cos x}$, $\cot x = \dfrac{\cos x}{\sin x}$, $\sec x = \dfrac{1}{\cos x}$, $\csc x = \dfrac{1}{\sin x}$

Optional Theorem C: (The Derivative of tan *x*.)

Prove: $(\tan x)' = \sec^2 x$ (i.e. $\dfrac{d}{dx}\tan x = \sec^2 x$).

Proof:

{This proof <u>was</u> asked on a U of M exam once; perhaps it will be asked again.}

{Make sure you have memorized the following identity: $\sin^2 x + \cos^2 x = 1$}

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' \rightarrow \frac{T'B - TB'}{B^2}$$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2}$$

$$= \frac{\overbrace{\cos^2 x + \sin^2 x}^{=1,\text{ by identity}}}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x \quad \underline{\text{proven}}$$

Optional Theorem D: (The Derivative of cot *x*.)

Prove: $(\cot x)' = -\csc^2 x$ (i.e. $\dfrac{d}{dx}\cot x = -\csc^2 x$).

Proof:

{This proof has <u>never</u> been asked on a U of M exam; perhaps it will one day.}

{Make sure you have memorized the following identity: $\sin^2 x + \cos^2 x = 1$}

$$(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' \rightarrow \frac{T'B - TB'}{B^2}$$

$$= \frac{(-\sin x)(\sin x) - (\cos x)(\cos x)}{(\sin x)^2} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= \frac{-\overbrace{\left(\sin^2 x + \cos^2 x\right)}^{=1,\text{ by identity}}}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x \quad \underline{\text{proven}}$$

Optional Theorem E: **(The Derivative of sec *x*.)**

Prove: $(\sec x)' = \sec x \tan x$ (i.e. $\frac{d}{dx}\sec x = \sec x \tan x$).

Proof:

{This proof has never been asked on a U of M exam; perhaps it will one day.}

$$(\sec x)' = \left(\frac{1}{\cos x}\right)' \rightarrow \frac{T'B - TB'}{B^2}$$
$$= \frac{(0)(\cos x) - (1)(-\sin x)}{(\cos x)^2} = \frac{\sin x}{\cos^2 x}$$
$$= \frac{\sin x}{\cos x \cdot \cos x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x \quad \underline{\text{proven}}$$

Optional Theorem F: **(The Derivative of csc *x*.)**

Prove: $(\csc x)' = -\csc x \cot x$ (i.e. $\frac{d}{dx}\csc x = -\csc x \cot x$).

Proof:

{This proof has never been asked on a U of M exam; perhaps it will one day.}

$$(\csc x)' = \left(\frac{1}{\sin x}\right)' \rightarrow \frac{T'B - TB'}{B^2}$$
$$= \frac{(0)(\sin x) - (1)(\cos x)}{(\sin x)^2} = \frac{-\cos x}{\sin^2 x}$$
$$= \frac{-\cos x}{\sin x \cdot \sin x} = \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x \quad \underline{\text{proven}}$$

Lesson 6: Implicit Differentiation

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

1. Find $\frac{dy}{dx}$ in terms of x and y for the curve $4x^3y^2 - 3xy = 5x + 2y$.

2. Given that y is a function of x such that $y\sin\left(\sqrt{2}x\right) + 2x^2\tan y^3 = \pi^3$

find $\frac{dy}{dx}$ in terms of x and y for the curve.

3. Find an equation of the tangent line to the curve $x\sqrt{y} + y\sqrt{x} = 6$ at the point (4, 1).

4. Find $\frac{d^2y}{dx^2}$ in terms of x and y for $x^3 - y^2 + y^3 = x$. Do not simplify.

Lesson 7: Related Rates

In a related rates problem, you are given one or more rates and are asked to find an unknown rate (see "Step 2" below for how you recognize a rate). The question is generally like, "How fast is <blank> changing?"

Step 1: Draw a diagram and label its dimensions. Be sure you distinguish between ***instants*** and ***constants***. For example, if you are told the width is 10 cm but it is changing at 2 cm/s, do not put 10 in the diagram, use a letter like w since the width is varying. Constants can certainly be labelled on the diagram.

To deal with *instants* draw a second diagram labelling all the values that are only true for that instant in time and use that diagram to help compute unknowns.

Step 2: List all the rates in the problem as "$\frac{d}{dt}$" statements. Rates can be identified by their *rate units* (blank *per* blank). For example, if we are told the *width* is increasing at 2 cm/sec, we can state $\frac{dw}{dt} = 2$. Be sure you also list the rate you want to find in the problem.

An important tip is that the rates that will be relevant in the problem will use the variables you labelled in your original diagram. For example, if your diagram has dimensions labelled x and y, then you will expect the rates $\frac{dx}{dt}$ and $\frac{dy}{dt}$ to be either given or asked for. The only rates that may be involved in a problem that do not relate to a labelled variable are $\frac{dV}{dt}$, the rate of change of the *volume* (if *cubic* units are given, such as m^3, cm^3, ft^3, etc.) and $\frac{dA}{dt}$, the rate of change of the *area* (if *square* units are given, such as m^2, cm^2, ft^2, etc.).

Step 3: State the equation relevant to the problem, using your diagram to help. Perhaps it is a volume formula; or area formula; or, for right triangles, the Pythagorean Theorem $\left(a^2 + b^2 = c^2\right)$, etc. As long as all the variables in the equation have been mentioned in your list of rates, you can proceed to the next step.

Step 4: "$\frac{d}{dt}$" both sides of the equation.

Step 5: Sub in all the given instants you noted on your second diagram and all the given rates you listed in Step 2, and solve the unknown rate. Be sure to write a sentence stating your answer to their question, including the proper units.

Memorize these useful Formulas and Facts:

Perimeter: Perimeter is the distance around a shape (how far you would walk if you walked around the shape). Perimeter and circumference are the same thing. Circumference is the term generally used for curved shapes. **In general, to find the perimeter of an object you simply add up the lengths of all of its sides.**

Surface Area: In general, to find the surface area of an object you simply compute the area of all of its separate surfaces and add them up.

Triangles: $Area = \frac{1}{2} base \times height$

Right Triangles: $a^2 + b^2 = c^2$ where c is the hypotenuse; a and b are the two legs.

Rectangles: $Area = length \times width$

Circles: $Circumference = C = 2\pi r$ $Area = A = \pi r^2$ where r is the radius.

Cylinders: *Volume = Area of the circular base × height*: $V = \pi r^2 h$

To find the *Surface Area* you must compute the area of the 2 circles that make up the top and bottom of the cylinder $\left(\pi r^2 \times 2 = 2\pi r^2\right)$ plus you must compute the area of the cylindrical wall. To visualize the area of the cylindrical wall, think of a paper towel tube. If you cut the tube down its length and flatten it out, you have a rectangle. The height of the rectangle is the height of the tube h. The length of the rectangle is the length of the circle you just flattened out (the length = the circumference of the circle = $2\pi r$). Therefore, the area of this rectangle is $2\pi r \times h = 2\pi rh$.

Surface Area = Area of the 2 circles + Area of the cylindrical wall: $A = 2\pi r^2 + 2\pi rh$

Trig Ratios: ***S**OH**C**AH**T**OA*: $\sin\theta = \frac{opposite}{hypotenuse}$ $\cos\theta = \frac{adjacent}{hypotenuse}$ $\tan\theta = \frac{opposite}{adjacent}$

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

1. The top of a ladder 10 metres long rests against a vertical wall. If the base of the ladder is being pulled away from the base of the wall at a constant rate of 1/4 metre per second, how fast is the top of the ladder slipping down the wall when it is 6 metres above the base of the wall?

2. At 1:00 PM ship A is 6 km due north of ship B. If ship A is sailing west at a rate of 10 km/h and ship B is sailing south at 12 km/h, find the rate at which the distance between the two ships is changing at 1:30 PM.

3. A water tank has the shape of an inverted circular cone with base diameter of 6 metres and height 4 metres. If water is being pumped into the tank at the rate of 2 m^3/minute, find the rate at which the water level is rising when the water is 3 metres deep. [Hint: the volume of a cone is $\frac{1}{3}\pi r^2 h$.]

4. A cylinder is being stretched in such a way that its volume remains constant while its radius decreases at 2 cm/min. Does the surface area remain constant? If not, how fast is the surface area changing at a moment when the radius is 10 cm and the height is 8 cm? [Hint: the volume of a cylinder is $\pi r^2 h$, while the *lateral* surface area is $2\pi rh$.]

5. A 5 foot tall woman is walking towards a 15 foot tall lamppost at 2 feet per second.

(a) When she is 6 feet away from the post, how fast is her shadow shrinking?

(b) How fast is the top of the head of her shadow moving at this time?

6. A lighthouse is located on a small island 2 km from the nearest point A on a long, straight shoreline. If the lighthouse lamp rotates at 3 revolutions per minute, how fast is the illuminated spot P on the shoreline moving along the shoreline at the moment when P is 4 km from A?