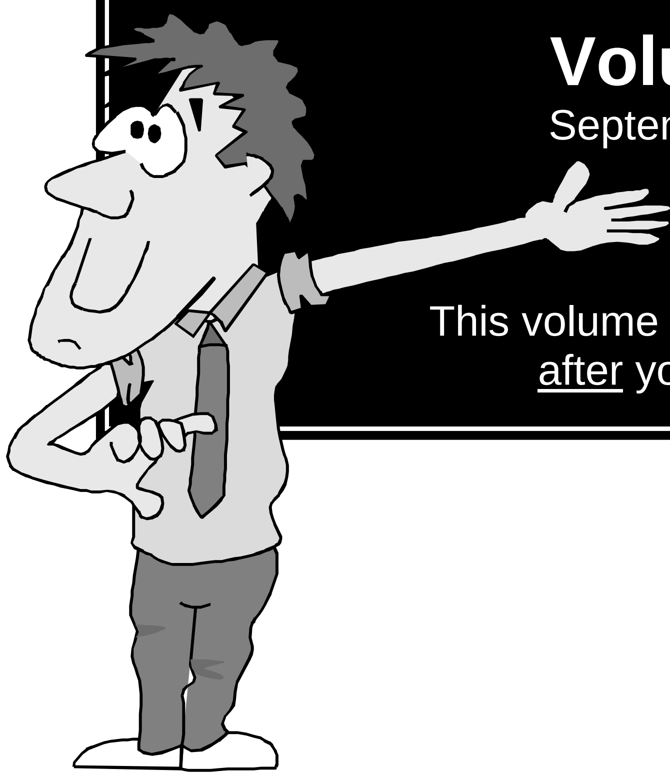
Grant's Tutoring

CALCULUS
for MANAGEMENT
Volume 2 of 2
September 2011 edition



This volume covers the topics taught after your midterm exam.

Learn What You Need to Know
Know What You Need to Learn

While studying this book, why not hear Grant explain it to you? Contact Grant for info about purchasing **Grant's Audio Lectures**. Some concepts make better sense when you hear them explained.

Better still, see Grant explain the key concepts in person. Sign up for **Grant's Weekly Tutoring** or attend **Grant's Exam Prep Seminars.** Text or Grant (204) 489-2884 or go to **www.grantstutoring.com** to find out more about all of Grant's services. **Seminar Dates will be finalized no later than Sep. 25 for first term and Jan. 25 for second term.**

HOW TO USE THIS BOOK

I have broken the course up into lessons. Study each lesson until you can do all of my lecture problems from start to finish without any help. Then do the Practise Problems for that lesson. If you are able to solve all the Practise Problems I have given you, then you should have nothing to fear about your Midterm or Final Exam.

I have presented the course in what I consider to be the most logical order. Although my books are designed to follow the course syllabus, it is possible your prof will teach the course in a different order or omit a topic. It is also possible he/she will introduce a topic I do not cover. **Make sure you are attending your class regularly! Stay current with the material, and be aware of what topics are on your exam. Never forget, it is your prof that decides what will be on the exam, so pay attention.**

If you have any questions or difficulties while studying this book, or if you believe you have found a mistake, do not hesitate to contact me. My phone number and website are noted at the bottom of every page in this book. "Grant's Tutoring" is also in the phone book. **I welcome your input and questions.**

Wishing you much success,

Grant Skene

Owner of Grant's Tutoring

Formulas and Definitions to Memorize

Point-Slope form of a Line: $y - y_0 = m(x - x_0)$

Vertex of a Parabola: For all parabolas $y = ax^2 + bx + c$, the vertex is at $x = \frac{-b}{2a}$.

Compound Interest Formula: $A = P\left(1 + \frac{r}{m}\right)^{mt}$

Continuous Compounding Formula: $A = Pe^{rt}$

Exponential Growth or Decay Formula: $Q = Q_0 a^t$

The Definition of Continuity: $f(x)$ is continuous at $x=a$ if and only if $\lim_{x \to a} f(x) = f(a)$.

The Definition of Derivative: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

The Power Rule: $(x^n)' = nx^{n-1}$

The Product Rule: $(f \cdot g)' = f'g + fg'$

The Quotient Rule: $\left(\frac{T}{B}\right)' = \frac{T'B - TB'}{B^2}$

The Chain Rule: $(f(u))' = f'(u) \cdot u'$

The Chain Rule Version of Power Rule: $(u^n)' = nu^{n-1} \cdot u'$

Derivatives of Exponential and Logarithmic Functions:

$$\left(e^u\right)' = e^u \cdot u' \qquad \left(\ln u\right)' = \frac{u'}{u}$$

$$\left(a^u\right)' = a^u \cdot u' \cdot \ln a \qquad \left(\log_a u\right)' = \frac{u'}{(u)\ln a}$$

The Fundamental Theorem of Calculus:

$$\int_a^b f'(x)\,dx = f(x)\Big|_a^b = f(b) - f(a)$$

$$\left(\int_a^b f'(x)\,dx \text{ is the change in } f(x) \text{ from } x = a \text{ to } x = b\right)$$

Antiderivative Formulas:

$$\int K\,dx = Kx + C$$

e.g. $\int 2\,dx = 2x + C$

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$$

e.g. $\int x^3\,dx = \frac{1}{4}x^4 + C$

e.g. $\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{2}{3}x^{3/2} + C$

e.g. $\int \frac{1}{x^2}\,dx = \int x^{-2}dx = \frac{1}{-1}x^{-1} = -x^{-1} + C$

$\boxed{\int \frac{1}{x}\,dx = \ln|x| + C}$ → Watch for this guy! Note: $\int x^{-1}dx = \int \frac{1}{x}\,dx = \ln|x| + C$

$$\int e^x\,dx = e^x + C$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

e.g. $\int e^{5x}\,dx = \frac{1}{5}e^{5x} + C$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

e.g. $\int 10^x\,dx = \frac{10^x}{\ln 10} + C$

Lesson 8: Log and Exponential Derivatives

MEMORIZE THE FOLLOWING:

e is a constant $(e = 2.718...)$ and "ln" is the log with base e:

$\boxed{\ln 1 = 0}$ and $\boxed{\ln e = 1}$

The 3 Log Laws:

$$\log_a(mn) = \log_a m + \log_a n \quad \rightarrow \quad \ln(mn) = \ln m + \ln n$$

$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n \quad \rightarrow \quad \ln\left(\frac{m}{n}\right) = \ln m - \ln n$$

$$\log_a\left(m^p\right) = p \cdot \log_a m \quad \rightarrow \quad \ln\left(m^p\right) = p \cdot \ln m$$

Derivatives of Exponential and Logarithmic Functions:

$$\left(e^u\right)' = e^u \cdot u'$$

$$\left(a^u\right)' = a^u \cdot u' \cdot \ln a$$

$$\left(\ln u\right)' = \frac{u'}{u}$$

Note that $\left(\ln|u|\right)' = \dfrac{u'}{u}$ also.

$$\left(\log_a u\right)' = \frac{u'}{(u)\ln a}$$

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

1. Find $\frac{dy}{dx}$ for the following. You need not simplify your answer.

(a) $y=\left(x^3-7x^2+10\right)e^{x^5+4}$

(b) $y=3x^5\ln\left(x-x^4\right)$

(c) $y=\dfrac{e^{x^2+3}}{4\ln x+x^2}$

(d) $y=\left(e^x+7\right)^3\ln\left(\dfrac{5}{x}+x^2\right)$

(e) $y=\ln\left|x^{10}-3x^3+e^5\right|$

(f) $y=\dfrac{\sqrt{\ln x}+2}{x^3+2x+1}$

(g) $y=5^{3x}\cdot x^5+\log_3\left(1+x^2\right)$

(h) $y=x^{\pi}+\pi^{x}+e^{\pi}$

(i) $y=\ln\left(x^2+1\right)+\log\left(x^3+1\right)$

(j) $y=2^{x^2-x}\log_2\left(x^3+7x\right)$

Lesson 9: Curve-Sketching

On scrap paper, set up a **Table-of-Values** where you will enter all relevant values of x which show up during your work. Essentially, all **Top Zeroes** will be put on this table. Always put $x=0$ on the table immediately which will give you your y-intercept. When you add a number to your table, state what it is (*critical point, local max, inflection point, etc.*) and, in the case of *critical* and *singular* points, sketch what the point looks like. Do not record asymptotes on the table, record them next to the table.

Step 1: Gather information from $f(x)$, the function itself.

- The **Top Zeroes** are the ***x*-intercepts** (put these on your Table-of-Values!); the **Bottom Zeroes** are the **Vertical Asymptotes**.
- Check for **Horizontal Asymptotes** by computing the limits as $x \to \pm\infty$.

Step 2: Gather information from $f'(x)$, the first derivative.

- The **Top Zeroes** are the **Critical Points (CP)** (put these on your Table-of-Values!); the **Bottom Zeroes** are probably just the **Vertical Asymptotes** you found earlier, if not, they are **Singular Points (SP)** (which, so far, have never shown up on a test!)
- Make a **sign diagram** for $f'(x)$. The sign diagram identifies where $f(x)$ is **increasing** and where $f(x)$ is **decreasing**.
- Looking at the sign diagram, observe the shape of your critical points and note the shape on your Table-of-Values. For each critical point note on the Table-of-Values whether it is a local maximum, local minimum, or neither.

Step 3: Gather information from $f''(x)$, the second derivative.

- The **Top Zeroes** are the **Inflection Points (IP)** (put these on your Table-of-Values!); the **Bottom Zeroes** are probably just the **Vertical Asymptotes** you found earlier, if not, they are perhaps inflection points also.
- Make a **sign diagram** for $f''(x)$. The sign diagram identifies where $f(x)$ is **concave up** and where $f(x)$ is **concave down** and should confirm the inflection points.

Step 4: Now we are ready to sketch the curve.

- Scan your Table-of-Values and your asymptotes and mark the relevant x and y values on your axes.
- Plot all the points and draw in the asymptotes you found.
- Draw through points with nice smooth arcs, but never draw through a point of inflection. Arc diagonally towards a point of inflection then stop, lift your pen from the page and be sure to change your concavity before you continue to draw the curve.
- If there is a Vertical Asymptote, make sure there is at least one point plotted on each side of the asymptote, adding a random x value to your Table-of-Values if necessary. Remember, as a curve nears a Vertical Asymptote it must either swoop up towards $+\infty$ or swoop down towards $-\infty$.
- If there is a Horizontal Asymptote, the far left side of the graph must begin *parallel* to the asymptote and the far right side of the graph must end *parallel* to the asymptote. Anything can happen in the middle; a curve can even cross a horizontal asymptote.

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

1. Make a sign diagram for the following expressions:

(a) $\dfrac{(x^2-4)^2(x-3)^3}{x^2+1}$ **(b)** $\dfrac{(x-1)^2(2x-5)^3(x+2)^{3/5}}{(x+4)^{4/3}}$

2. Find the horizontal and vertical asymptotes for the following functions:

(a) $y=\dfrac{x^2+3x-5}{x^2-5x-6}$ **(b)** $y=\dfrac{(3x-1)^2(2x+1)}{x(x+2)(x-1)}$ **(c)** $y=\dfrac{3x^2+5x-2}{x^2-4}$

3. For the functions below:

(i) Find the domain of the function.

(ii) Discuss the symmetry.

(iii) Find the vertical and horizontal asymptotes

(iv) Find the local extremes and give the intervals where the function is increasing and decreasing.

(v) Find the points of inflection and the intervals where the function is concave up and concave down.

(vi) Sketch the graph of the function, labelling all relevant information.

(a) $y=2x(x+4)^3$, you are given $y'=8(x+1)(x+4)^2$, $y''=24(x+2)(x+4)$

(b) $y=\dfrac{3x^2-1}{x^3}$, you are given $y'=\dfrac{3-3x^2}{x^4}$, $y''=\dfrac{6x^2-12}{x^5}$

(c) $f(x)=4x+\dfrac{1}{x-1}$, you are given $f'(x)=\dfrac{(2x-1)(2x-3)}{(x-1)^2}$, $f''(x)=\dfrac{2}{(x-1)^3}$

(d) $y=xe^{-x^2/2}$, you are given $y'=(1-x^2)e^{-x^2/2}$, $y''=x(x^2-3)e^{-x^2/2}$

(e) $y=(x^2-1)^{2/3}$, you are given $y'=\dfrac{4x}{3(x^2-1)^{1/3}}$, $y''=\dfrac{4(x^2-3)}{9(x^2-1)^{4/3}}$

4. Given $f(x)=\dfrac{x^5}{\left(x^2-1\right)^2}$, $f'(x)=\dfrac{x^6-5x^4}{\left(x^2-1\right)^3}$ and $f''(x)=\dfrac{4x^5+20x^3}{\left(x^2-1\right)^4}$.

(a) Find the domain of $f(x)$, all zeroes of f and intervals where $f > 0$ and $f < 0$.

(b) Find all vertical and horizontal asymptotes.

(c) Discuss and verify any symmetries of $f(x)$.

(d) Find intervals where f is increasing, where f is decreasing, and all local extrema.

(e) Find intervals where f is concave up, where f is concave down, and all inflection points.

(f) Sketch the curve of $y=f(x)$, illustrating the above information.

5. For the function $f(x)=x^4-8x^2-3$, find its absolute maximum and absolute minimum values on the interval [–2, 1] and locate the points where they occur.

Lesson 10: Max/Min Word Problems

A problem of this sort is trying to maximize or minimize some quantity Q. For example, you may be asked to find the greatest *volume* (Q is the *volume*, which you wish to maximize); the least *cost* (Q is the *cost*, which you wish to minimize); and so on.

Step 1: **Draw a diagram and label its dimensions.** Generally, the diagram will require two variables, say x and y. If it needs less, great; if it needs more, you missed a way of using only two variables. Use the diagram to help make your equations.

Step 2: Since there are *two variables* generally, you will need *two equations*. I call them the ***Q* equation** and the **Constraint equation**. It does not matter which equation you come up with first.

- The ***Q* equation** relates x and y to Q, the quantity being maximized or minimized, and so will have the form $Q = f(x, y)$ (*i.e.* Q = a formula with x and y in it). This formula is usually obvious, like a known volume or area formula.
- The **Constraint equation** relates x and y to a *constant* that has been given in the problem, and so will have the form $\# = g(x, y)$ (*i.e.* # = a formula with x and y in it). This formula enables you to isolate y (or x, if that is easier) and substitute the result into the Q equation.

In general, we isolate *y* (or *x*) in the Constraint equation and sub that into the Q equation.

Step 3: Once you have substituted, you have $Q = f(x)$, a function of only one variable. Simplify the function in preparation for doing a derivative. Get rid of brackets and pull denominators up as negative exponents so that you can avoid using the Product and Quotient Rules as much as possible.

Step 4: Compute Q' and simplify it. Pull any negative exponents back down to the denominator, get a common denominator if necessary, and factor completely.

Step 5: Do a complete first derivative analysis as in curve-sketching.

- Find *Top Zeroes* (critical points) and *Bottom Zeroes* (assume these are vertical asymptotes).
- Make a sign diagram for Q'. If you are looking for a max, you can bet the local max is your answer; if you are looking for a min, you can bet the local min is your answer.
- If the sign diagram has more than 2 arrows on it, you have not proved the critical point is the answer you are looking for. In these cases, cut the sign diagram down by pointing out x's endpoints. Usually, you can declare $x > 0$, since x is a dimension.

In general, we declare, "*Q* is max (or min) at *x* = the critical point."

Step 6: Now reread the question and make sure you actually answer it!

- Typically, you will need to know both x and y (sub the x-value you found in Step 5 into the *Constraint* equation to get the y-value). You can then use those values to give them what they want.

Memorize these useful Formulas and Facts:

Perimeter: Perimeter is the distance around a shape (how far you would walk if you walked around the shape). Perimeter and circumference are the same thing. Circumference is the term generally used for curved shapes. **In general, to find the perimeter of an object you simply add up the lengths of all of its sides.**

Surface Area: In general, to find the surface area of an object you simply compute the area of all of its separate surfaces and add them up.

Triangles: $Area = \frac{1}{2} base \times height$

Right Triangles: $a^2 + b^2 = c^2$ where c is the hypotenuse; a and b are the two legs.

Rectangles: $Area = length \times width$

Circles: $Circumference = C = 2\pi r$ $\quad$ $Area = A = \pi r^2$

Cylinders: *Volume = Area of the circular base × height*: $V = \pi r^2 h$

To find the *Surface Area* you must compute the area of the 2 circles that make up the top and bottom of the cylinder $\left(\pi r^2 \times 2 = 2\pi r^2\right)$ plus you must compute the area of the cylindrical wall. To visualize the area of the cylindrical wall, think of a paper towel tube. If you cut the tube down its length and flatten it out, you have a rectangle. The height of the rectangle is the height of the tube h. The length of the rectangle is the length of the circle you just flattened out (the length = the circumference of the circle = $2\pi r$). Therefore, the area of this rectangle is $2\pi r \times h = 2\pi rh$.

Surface Area = Area of the 2 *circles + Area of the cylindrical wall*: $A = 2\pi r^2 + 2\pi rh$

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

1. The cost of producing q pieces of jewellery is $100+200q-20q^2+q^3/3$ dollars and each piece of jewellery will sell for $700. How many pieces should be sold to maximize the profit? Justify your answer.

2. Assume that the demand function for a certain product is $q=500e^{-0.02p}$ where p is the price in dollars and q is the number of items sold.

(a) Find the revenue function $R(p)$ as a function of price.

(b) Find the rate of change of the revenue when the price is $10. Should the price be increased?

(c) What price will maximize revenue? Justify your answer.

3. A rectangular enclosure is to be constructed having one side along an existing long wall and the other three sides fenced. If 100 metres of fence are available, what is the largest possible area for the enclosure?

4. A billboard is to contain 30 square metres of printed area with margins of 2 metres at top and bottom and 1 metre on each side. What outside dimensions will minimize the total area of the billboard?

5. A box is to be made from an 8 foot by 3 foot rectangular sheet of tin by cutting equal squares out of the four corners and bending up the resulting four flaps to make the sides of the box. (The box has no top.) What is the largest possible volume of the box?

6. A rectangular box with an open top and a square base is to have a volume of 320 cubic inches. If the material for the base costs 50 cents per square inch and the material for the four sides costs 40 cents per square inch, find the dimensions of the box that minimize the cost of the material from which it is made.

7. A rectangle has one vertex at the origin, one vertex on the positive x-axis, one vertex on the positive y-axis, and one vertex on the curve $f(x)=e^{-x}$. Of all such rectangles, find the dimensions of the one having the largest area.

8. A cylindrical tin can is to have a volume of 100π cubic centimetres. The top of the can costs twice as much per square centimetre than the bottom or wall. What radius will cost the least? [Hint: the volume of a cylinder is $\pi r^2 h$ while the *lateral* surface area is $2\pi rh$.]

9. Find the point on the graph of $x - y^{3/2} = 1$ that is closest to the point (1, 4).

10. Find the largest possible area for an isosceles triangle if it has a perimeter of 2 metres.

11. A submarine is travelling due east at 30 km/hour and heading straight for a point *P*. A battleship is travelling due south at 20 km/hour and heading for the same point *P*. At 4:00 am, their distances from *P* are 210 km for the submarine and 140 km for the battleship. At what time will they be closest to each other?

12. A lighthouse *L* is located on a small island 5 km north of a point *A* on a straight east-west shoreline. A cable is to be laid from *L* to point *B* on the shoreline 10 km east of *A*. The cable will be laid through the water in a straight line from *L* to a point *C* on the shoreline between *A* and *B* and from there to *B* along the shoreline. If the part of the cable lying in the water costs $5000/km and the part along the shoreline costs $3000/km, where should *C* be chosen to minimize the total cost of the cable?

LESSON 11: Antiderivatives (Integrals)

The Fundamental Theorem of Calculus:

$$\int_a^b f'(x)\,dx = f(x)\Big|_a^b = f(b) - f(a)$$

$$\left(\int_a^b f'(x)\,dx \text{ is the change in } f(x) \text{ from } x = a \text{ to } x = b\right)$$

Antiderivative Formulas:

$$\int K\,dx = Kx + C$$

e.g. $\int 2\,dx = 2x + C$

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$$

e.g. $\int x^3\,dx = \frac{1}{4}x^4 + C$

e.g. $\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{2}{3}x^{3/2} + C$

e.g. $\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{1}{-1}x^{-1} = -x^{-1} + C$

$\boxed{\int \frac{1}{x}\,dx = \ln|x| + C}$ → Watch for this guy! Note: $\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$

$$\int e^x\,dx = e^x + C$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

e.g. $\int e^{5x}\,dx = \frac{1}{5}e^{5x} + C$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

e.g. $\int 10^x\,dx = \frac{10^x}{\ln 10} + C$

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

1. Solve the following definite and indefinite integrals:

(a) $\int \left(6x^5 + 12x^2 - 8\right) dx$

(b) $\int \left(e^{3x} + \frac{2}{\sqrt[5]{x}}\right) dx$

(c) $\int \left(\frac{1}{x} + \frac{3}{2x^2} + e^{2x} - \sqrt[3]{x^2}\right) dx$

(d) $\int \left(2 + \frac{5}{x}\right) dx$

(e) $\int \left(x^2 + 1\right)\sqrt{x}\, dx$

(f) $\int \frac{e^{-5x} + x^5}{3}\, dx$

(g) $\int_0^1 \left(\sqrt{x} - 8x^3\right) dx$

(h) $\int_1^2 (2x-1)^2\, dx$

2. Find $f(x)$ for the following:

(a) $f'(x) = e - \frac{1}{2x}, \quad f(1) = e$

(b) $f'(x) = \frac{x^2 + x + 1}{x}, \quad f(-1) = 2$

(c) $f''(x) = (3x-1)^2, \quad f(2) = 0, \quad f'(1) = 0$

3. If the graph of $y = f(x)$ passes through the point (1, –2), and the slope of its tangent line at any point $(x, f(x))$ is given by the equation $12x^2 - 24x + 1$, find $f(x)$.

4. You know that $\int_0^5 f(x)dx = 25$ and $\int_0^2 f(x)dx = 5$, what is $\int_2^5 f(x)dx$.

5. Find the area bounded by the curve $y = 4x^3 + 4x$, the x-axis, and the vertical lines x=0 and x=2.

6. Find the area of the region of the plane lying below the parabola $y = 1 - x^2$ and above the x-axis.

7. Find the area of the region bounded by the curve $y = x^2 - 4x - 5$, the x-axis, and the lines x = 3 and x = 6. (Hint: draw a picture.).

8. **(a)** If the revenue function is $R(q)$, where q is the number of widgets and R is in dollars, explain the significance of $\int_{20}^{80} R'(q)\, dq = 1000$.

(b) If the revenue is $1200 when 20 widgets are sold, what will be the revenue for selling 80 widgets?

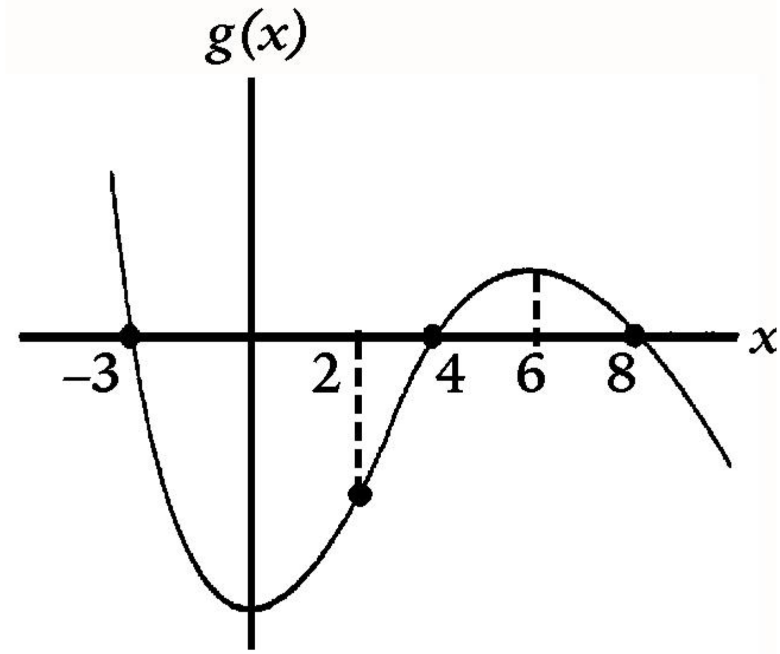
9. The marginal cost in producing q widgets is $6q^2 - 2q + 10$ hundred dollars per widget. There is a fixed cost of $400 to produce widgets.

(a) What is the rate of change of the cost when we are making 10 widgets?

(b) Express C, the cost, as a function of q.

(c) How much will it cost to make 10 widgets?

10. For the graph of $g(x)$ as shown determine whether the following quantities are positive, negative or zero.



(a) $g(-3)$

(b) $g'(-3)$

(c) $g(0)$

(d) $g'(0)$

(e) $g(2)$

(f) $g'(2)$

(g) $\dfrac{g(8)-g(2)}{8-2}$

(h) $\dfrac{g(8)-g(6)}{2}$

(i) $\lim\limits_{h\to 0}\dfrac{g(4+h)-g(4)}{h}$

(j) $\lim\limits_{h\to 0}\dfrac{g(6+h)-g(6)}{h}$

(k) $g''(-3)$

(l) $g''(0)$

(m) $g''(2)$

(n) $g''(4)$

(o) $\int_4^8 g(x)dx$

(p) $\int_{-3}^8 g(x)dx$

LESSON 12: Partial Derivatives

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

1. Compute the first order partial derivatives for the following.

(a) $f(x,y) = 2x^3 e^{x^2+y^5}$

(b) $f(x,y) = \dfrac{9+4x}{6y - x^2 - y^2}$

2. Compute the first and second order partial derivatives for the following. Which is to say, compute f_x, f_y, f_{xx}, f_{xy}, f_{yx}, and f_{yy}.

(a) $f(x,y) = x^3 + xy + y^2 + 6$

(b) $f(x,y) = x^2 + 5x^3y^4 + 3y^5$