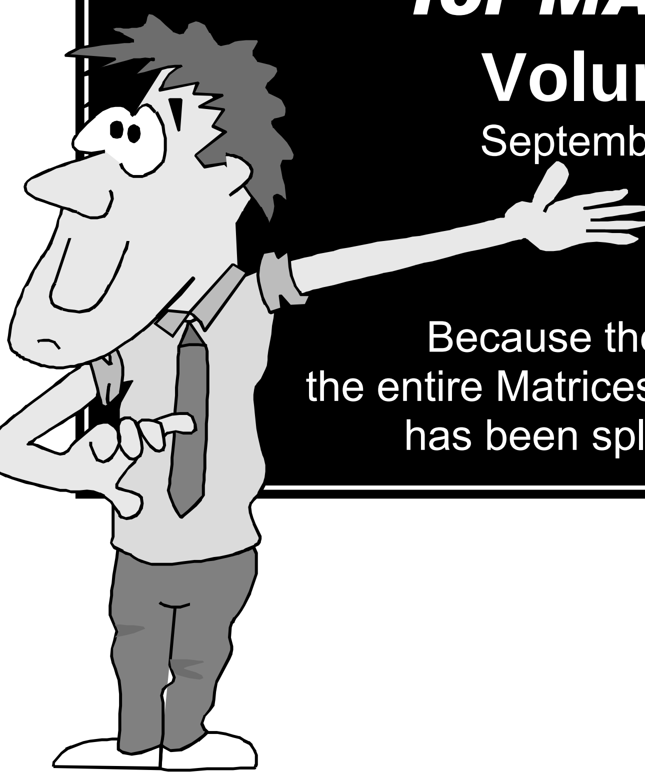
Grant's Tutoring

MATRICES

for MANAGEMENT

Volume 1 of 2

September 2014 edition



Because the book is so large,
the entire Matrices for Management course
has been split into two volumes.

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I have broken the course up into lessons. Study each lesson until you can do all of my lecture problems from start to finish without any help. Then do the Practise Problems for that lesson. If you are able to solve all the Practise Problems I have given you, then you should have nothing to fear about your Midterm or Final Exam.

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If you have any questions or difficulties while studying this book, or if you believe you have found a mistake, do not hesitate to contact me. My phone number and website are noted at the bottom of every page in this book. "Grant's Tutoring" is also in the phone book. **I welcome your input and questions.**

Wishing you much success,

Grant Skene

Owner of Grant's Tutoring

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(These Lessons are in Volume 2)

LESSON 1: SYSTEMS OF LINEAR EQUATIONS

Warning: *The following lesson is intended as a review of and introduction to basic concepts of linear systems. I think you will find this lesson helpful to give you context for this course, but it is quite possible you will never be tested on the material and methods taught here. It will, however, give you the necessary background to understand and appreciate the later lessons.*

A linear equation has one or more variables (like x or y) raised to the power of 1. For example, $2x+3y=6$ is a linear equation; both x and y have understood powers of 1. An equation is nonlinear if it has any variables raised to other powers (like x^2 or y^3); if the variables are under roots (like $\sqrt{x}$ or $\sqrt[3]{y}$); if the variables are in denominators (like $\frac{6}{x}$); if the variables are part of a transcendental function (examples of transcendental functions are trigonometric functions like $\sin x$; exponential functions like e^x or 2^y; logarithmic functions like $\ln x$ or $\log x$). **A term that contains more than one variable is nonlinear** ($6xy$ is a nonlinear term because it has two variables multiplying together even though both of those variable are raised to the understood power of 1). **The coefficients (the numbers in front of the variables) can come in all shapes and sizes, however. There is also no limit on the amount of variables in a linear equation, so long as the variables are strictly and only raised to the power of 1.**

Here are some examples:

- $2x+3y+4z=7$ is a **linear** equation. Although, there are three variables (x, y, z), they are all raised to the power of 1, and so are linear.
- $\sqrt{3}\,x+\frac{2}{5}y=12$ is a **linear** equation. Even though it has weird coefficients like "$\sqrt{3}$" and "$\frac{2}{5}$", its variables are raised to the power of 1 ("x" and "y") making it linear.
- $3x-4\sqrt{y}=7$ is a **nonlinear** equation because of the "$\sqrt{y}$" term.
- $3x-4xy+5y=10$ is a **nonlinear** equation because of the "xy" term.
- $4x^2-5x+4y=8$ is a **nonlinear** equation because of the "x^2" term.
- $6\sin x+3\cos y-\log_3 x=10$ is a **nonlinear** equation. You've got to be kidding me! It's not even close; it has trigonometric and logarithmic functions in it.

GRAPHING A LINEAR EQUATION

The fundamental linear equation has two variables (we usually designate them by x and y, but any symbols could be used). **Linear equations are so-called because they graph as a line.** The **standard form** of a linear equation is $\boldsymbol{ax+by=c}$ where a, b and c are any real number constants. For example, $2x+3y=6$ is a linear equation in standard form.

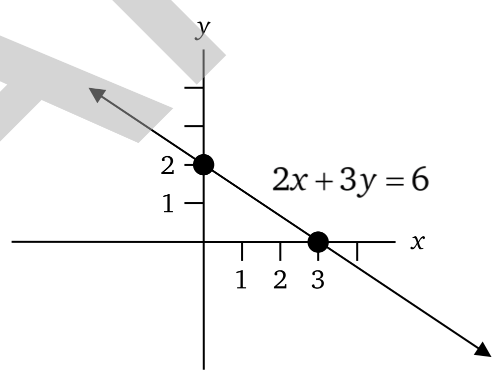
To quickly graph a line, we need only plot two points. The easiest points to plot are the intercepts. **To get the *y*-intercept, sub in *x* = 0. To get the *x*-intercept, sub in *y* = 0.** If I wanted to graph $2x+3y=6$, I would make a table-of-values like so:

x	y
0 sub $y = 0$ into $2x+3y=6$ to solve x	sub $x = 0$ into $2x+3y=6$ to solve y 0

Therefore, the table of values for $2x+3y=6$ would be:

x	y
0	2
3	0

We could now plot these two points and draw a line through them to make our graph.

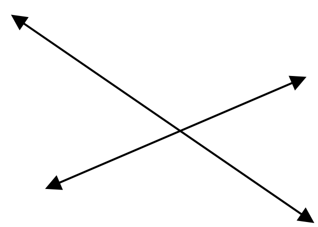


If you have two or more equations, you have a **system of equations**. The goal is to then find the solution or solutions that satisfy <u>all</u> the equations. **Geometrically speaking** (i.e. if we were looking at a graph of the system), **we are trying to find the intersection of the graphs**; the point or points where the separate graphs contact each other.

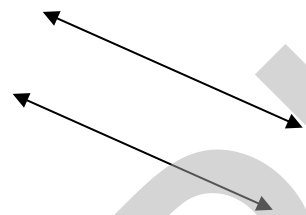
LINEAR SYSTEMS WITH TWO VARIABLES

Let's first focus on the most straightforward system of equations: two linear equations with two variables. Geometrically speaking, we have two lines and want to find where they intersect. There are three possibilities:

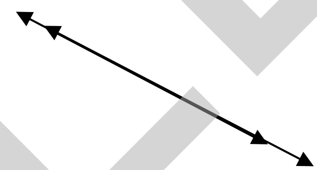
The two lines have a single point of intersection.



The two lines are parallel; they do not intersect at all.



The two lines are right on top of each other; they have infinite points of intersection.



1. Solve the system of equations below using the elimination method, and interpret the solution geometrically.

$$\begin{cases} 2x+3y=6 \\ 5x+2y=-7 \end{cases}$$

SOLUTION

In the elimination method we <u>add</u> the columns in such a way that one of the variables is <u>eliminated</u>.* Essentially, the terms to be eliminated must have identical coefficients, *but with the opposite sign*. We can multiply an equation by any number we want to accomplish this (just make sure you multiply both sides of the equation to maintain balance).

For no particular reason, I will eliminate the "y" terms (I could just as easily eliminate the "x" terms). I will multiply every term in the first equation by -2 to create a "$-6y$" term and multiply every term in the second equation by 3 to create a "$+6y$" term.

$$\begin{array}{lcccl} 2x+3y=6 & \to & \text{multiply by } -2 & \to & -4x-\cancel{6y}=-12 \\ 5x+2y=-7 & \to & \text{multiply by } 3 & \to & \underline{15x+\cancel{6y}=-21} \\ & & \text{Add the columns} & \to & 11x \qquad = -33 \end{array}$$

$$\boldsymbol{x=\frac{-33}{11}=-3}$$

* Some people prefer to <u>subtract</u> the columns to eliminate a variable. I strongly advise against this as many students often carelessly losing track of negative signs while performing the math.

Now that we have solved x, we can substitute this value back into either one of the original equations to solve y. I will sub it into the first equation, but I could just as easily use the second one (either equation better produce the same value for y, or we have definitely made a mistake).

Sub $\boldsymbol{x = -3}$ into $2x+3y=6$:

$$2(-3)+3y=6 \;\rightarrow\; -6+3y=6 \;\rightarrow\; 3y=12 \;\rightarrow\; \boldsymbol{y=\frac{12}{3}=4}$$

We have established the solution to this system is $\boldsymbol{x = -3, y = 4}$. Put another way, we have found both lines intersect at the point $(-3, 4)$.

We can check our answer by confirming $(-3, 4)$ satisfies both equations.

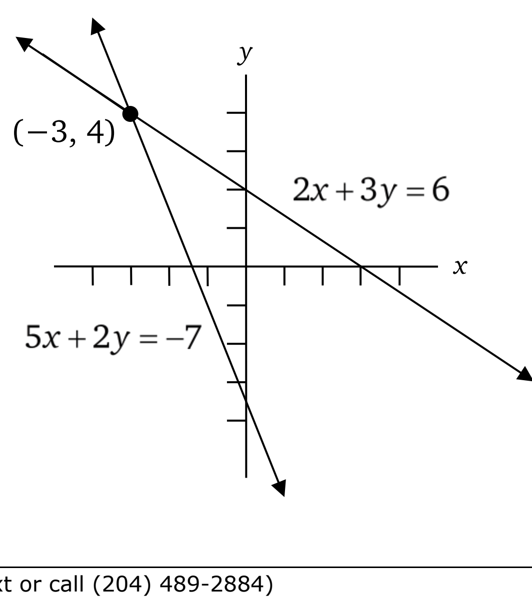
Subbing $(-3, 4)$ into $2x+3y=6$, we get $2(-3)+3(4)=6 \;\rightarrow\; -6+12=6 \;\rightarrow\; 6=6$ ✓

Subbing $(-3, 4)$ into $5x+2y=-7$, we get $5(-3)+2(4)=-7 \;\rightarrow\; -15+8=-7 \;\rightarrow\; -7=-7$ ✓

Thus, both lines pass through the point $(-3, 4)$.

The solution to this system of equations is $\boldsymbol{x = -3, y = 4}$. Interpreting this solution geometrically, we have discovered a graph of these two lines intersects at the point $\boldsymbol{(-3, 4)}$.

Although the question does not ask us to display the graphs, let's do so just to visualize what we mean by interpreting the solution geometrically. As our check confirmed, the two lines cross at the point $(-3, 4)$ verifying that is the one and only solution to this system of linear equations.



2. Solve the system of equations below using the elimination method, and interpret the solution geometrically.

$$\begin{cases} \dfrac{x}{4}+y=1 \\ \dfrac{x}{6}-\dfrac{5y}{3}=3 \end{cases}$$

SOLUTION

If they are nasty enough to put fractions in an equation, get rid of them! (The fractions, not the people who put them there.) Multiply the equation by the common denominator. The first equation has a denominator of 4, so I will multiply every term by 4 to get rid of it. The second equation has denominators of 6 and 3, so the common denominator is 6. I will multiply every term by 6 to get rid of them.

$$\frac{x}{4}+y=1 \rightarrow \text{multiply by 4} \rightarrow \frac{\cancel{4}x}{\cancel{4}}+4y=4 \rightarrow x+4y=4$$

$$\frac{x}{6}-\frac{5y}{3}=3 \rightarrow \text{multiply by 6} \rightarrow \frac{\cancel{6}x}{\cancel{6}}-\frac{{}^{2}\cancel{6}\times 5y}{\cancel{3}}=6\times 3 \rightarrow x-10y=18$$

Thus, the given system of equations is equivalent to the system: $\begin{cases} x+4y=4 \\ x-10y=18 \end{cases}$

$$\begin{array}{lcccr} x+4y=4 & \rightarrow & \text{leave it alone} & \rightarrow & \cancel{x}+4y=4 \\ x-10y=18 & \rightarrow & \text{multiply by } -1 & \rightarrow & -\cancel{x}+10y=-18 \\ \hline & & \text{Add the columns} & \rightarrow & 14y=-14 \end{array}$$

$$\boldsymbol{y=\frac{-14}{14}=-1}$$

Sub $\boldsymbol{y=-1}$ into either one of the two equations to get x. I will use $x+4y=4$:

$$x+4(-1)=4 \rightarrow x-4=4 \rightarrow \boldsymbol{x=8}$$

Sub (8, −1) into both of the original equations to check the answer:

$$\frac{x}{4}+y=1 \rightarrow \text{sub in } (8,-1) \rightarrow \frac{8}{4}+(-1)=1 \rightarrow 2-1=1 \rightarrow 1=1\ ✓$$

$$\frac{x}{6}-\frac{5y}{3}=3 \rightarrow \text{sub in } (8,-1) \rightarrow \frac{8}{6}-\frac{5(-1)}{3}=3 \rightarrow \frac{4}{3}+\frac{5}{3}=3 \rightarrow \frac{9}{3}=3\ ✓$$

The solution to this system of equations is $x = 8$, $y = -1$.
These two lines intersect at the point (8, −1).

3. Solve the system of equations below using the elimination method, and interpret the solution geometrically.

$$\begin{cases} 3x+4y=5 \\ 5x-2y=12 \end{cases}$$

SOLUTION

$$\begin{array}{lcccl} 3x+4y=5 & \rightarrow & \text{leave it alone} & \rightarrow & 3x+\cancel{4y}=5 \\ 5x-2y=12 & \rightarrow & \text{multiply by 2} & \rightarrow & 10x-\cancel{4y}=24 \\ & & \text{Add the columns} & \rightarrow & 13x \quad =29 \end{array}$$

$$\boldsymbol{x=\frac{29}{13}}$$

Tip: Rather than go through the ordeal of fraction math to solve *y* by substitution, go back to the original system and eliminate *x* this time.

$$\begin{array}{lcccl} 3x+4y=5 & \rightarrow & \text{multiply by 5} & \rightarrow & \cancel{15x}+20y=25 \\ 5x-2y=12 & \rightarrow & \text{multiply by } -3 & \rightarrow & -\cancel{15x}+6y=-36 \\ & & \text{Add the columns} & \rightarrow & 26y=-11 \end{array}$$

$$\boldsymbol{y=-\frac{11}{26}}$$

Check **(29/13, −11/26)** is the correct solution.*

Subbing (29/13, −11/26) into $3x+4y=5$, we get :

$$3\left(\frac{29}{13}\right)+4\left(-\frac{11}{26}\right)=5 \rightarrow \frac{87}{13}-\frac{\overset{22}{\cancel{44}}}{\underset{13}{\cancel{26}}}=5 \rightarrow \frac{87}{13}-\frac{22}{13}=5 \rightarrow \frac{65}{13}=5 \checkmark$$

Subbing (29/13, −11/26) into $5x-2y=12$, we get :

$$5\left(\frac{29}{13}\right)-2\left(-\frac{11}{26}\right)=12 \rightarrow \frac{145}{13}+\frac{\overset{11}{\cancel{22}}}{\underset{13}{\cancel{26}}}=12 \rightarrow \frac{145}{13}+\frac{11}{13}=12 \rightarrow \frac{156}{13}=12 \checkmark$$

The solution to this system of equations is *x* = 29/13, *y* = −11/26.
These two lines intersect at the point (29/13, −11/26).

* Never check your solutions to exam questions until you have completed the entire exam. **Don't waste time checking answers when you have other questions to do.** If you're right, you just wasted precious time proving it; if you are wrong, you don't want to know! Get the test finished first, then check if time allows.

> **4. Solve the system of equations below using the elimination method, and interpret the solution geometrically.**
>
> $$\begin{cases} \mathbf{2x+3y=6} \\ \mathbf{4x+6y=6} \end{cases}$$

SOLUTION

$$\begin{array}{lcccr} 2x+3y=6 & \rightarrow & \text{multiply by } -2 & \rightarrow & -\cancel{4x}-\cancel{6y}=-12 \\ 4x+6y=6 & \rightarrow & \text{leave it alone} & \rightarrow & \cancel{4x}+\cancel{6y}=\ \ 6 \\ \hline & & \text{Add the columns} & \rightarrow & 0=-6? \end{array}$$

Whoa! What happened here? Both variables got eliminated at the same time! That left us with just “0” on the left hand side of the equation after we added the columns. Specifically, we got “0 = −6”. This is clearly a false statement; 0 and −6 are not equal at all!

> **If, when performing the elimination method on a system of two linear equations with two variables, you end up eliminating both variables at the same time, there are two possibilities:**
>
> - **You end up with a false equation “0 = *k*” where *k* is a nonzero number. The false statement tells us there is no solution to the system; the lines must be parallel.**
> - **You end up with the true equation “0 = 0”. This true statement tells us there are infinite solutions to the system; the lines must be right on top of each other; any point on the first line will also be on the second line.**

> **There is no solution to this system of equations since 0 ≠ −6. Interpreting this solution geometrically, we have discovered the two lines are parallel and, therefore, do not intersect.**

Although the question does not ask us to display the graphs, let’s do so. As we can see on the next page, the two lines are indeed parallel.

Table of Values
$4x+6y=6$

x	y
0	1
3/2	0

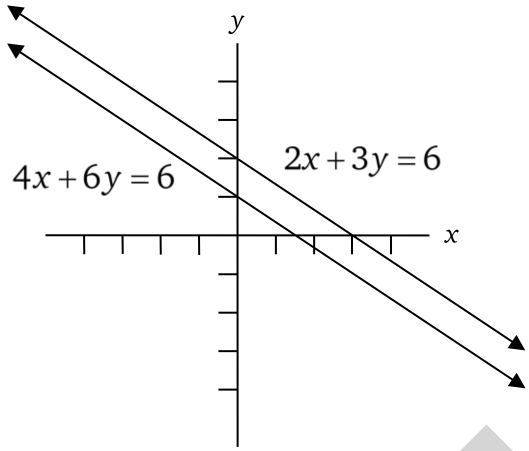


Table of Values
$2x+3y=6$

x	y
0	2
3	0

5. Solve the system of equations below using the elimination method, and interpret the solution geometrically.

$$\begin{cases} \boldsymbol{x-4y=4} \\ \boldsymbol{-2x+8y=-8} \end{cases}$$

SOLUTION

$$\begin{array}{rcccr} x-4y=4 & \rightarrow & \text{multiply by 2} & \rightarrow & \cancel{2x}-\cancel{8y}=8 \\ -2x+8y=-8 & \rightarrow & \text{leave it alone} & \rightarrow & \underline{\cancel{-2x}+\cancel{8y}=-8} \\ & & \text{Add the columns} & \rightarrow & 0\ =\ 0 \end{array}$$

Since the elimination has resulted in "0 = 0", we discover this system has infinite solutions. In fact, we have discovered these two equations are actually multiples of each other and, therefore, really the same line.

Table of Values
$x-4y=4$

x	y
0	−1
4	0

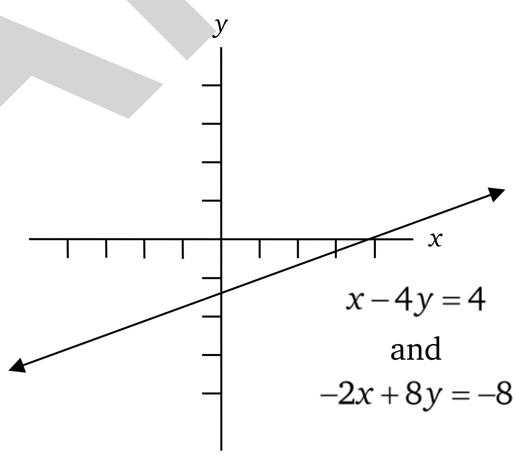


Table of Values
$-2x+8y=-8$

x	y
0	−1
4	0

Just because there are infinite solutions does not mean everything is a solution. **Infinite does not mean *everything*.** For example, as we can see on the graph of this system above, (0, 0) is not a solution to this system since it is not on the lines. Only points on the lines are solutions to this system. Admittedly, there are infinite points on the lines, but that is nothing compared to the amount of points not on the lines.

When there are infinite solutions, we must tell people what all the solutions are. They have to be clear which points are solutions and which are not. One way is to pick whichever of the two equations you like (since they are describing the same line anyway), and tell them the solutions are all the points on that line. So, I could say, the solution to this system is the infinite number of points on the line $x-4y=4$. But that's not good enough. Especially by the time we get to Lesson 2 and encounter larger, more complicated systems of linear equations, we need a more thorough way of describing the infinite solutions.

We introduce a parameter and state all the variables in terms of it. A parameter is a free variable, free to be any real number. The most common letter we use to represent a parameter is t; another commonly used symbol is s, but you could really use any letter you want. This problem has two variables, x and y. We can pick whichever one we want and simply let it equal t. I will let $y = t$, which is to say, y can be any real number; y has infinite values. (I could just as easily let $x = t$.) We know all the solutions satisfy the equation $x-4y=4$.

Sub $\boldsymbol{y = t}$ into $x-4y=4$ and solve for x: $x-4t=4 \;\rightarrow\; \boldsymbol{x=4+4t}$

We now have a "recipe" for all the solutions to the system: $\boldsymbol{x = 4 + 4t,\ y = t}$. Any real number we choose for t will produce a solution to the system. For example, if we let $t = 0$, we get $x = 4$, $y = 0$. If we let $t = 3$, we get $x = 16$, $y = 3$. There are infinite choices for t (we could let $t = -7$, $t = 1/3$, $t = \sqrt{5}$, any real number we can think of), producing infinite solutions to this system.

There are infinite solutions to this system of equations since 0 = 0. The solutions are $\boldsymbol{x = 4 + 4t,\ y = t}$ where $\boldsymbol{t}$ is any real number. Interpreting this solution geometrically, we have discovered the two lines are, in fact, the same line. All points in the form $\boldsymbol{(4 + 4t,\ t)}$ are solutions to this system.[*]

[*] If you let $x = s$ instead (I could have used t again, but I don't want this answer to be confused with the answer above), and sub that into $x-4y=4$, we get $s-4y=4 \;\rightarrow\; -4y=4-s \;\rightarrow y=\frac{4-s}{-4}=\frac{4}{-4}+\frac{-s}{-4} \;\rightarrow\; y=-1-\frac{s}{4}$

Thus, $x = s$, $y=-1-\frac{s}{4}$ or $\left(s, -1-\frac{s}{4}\right)$ is an equivalent answer (it generates all the same points).

LINEAR SYSTEMS WITH THREE VARIABLES

If you have a linear equation with three variables, $ax+by+cz=d$, you actually have a **plane** rather than a line. For example, $x+2y+3z=6$ is a plane in standard form. A plane is a flat, two-dimensional surface; i.e. it has length and width. A table-top is a plane; the floor is a plane; the walls are planes; the slanted roof on the outside of a typical home is a plane. The equation of a plane is still considered a linear equation because all its variables are raised to the power of 1.[*]

We are now dealing with three-dimensional coordinate geometry. Assuming you are in a nice ordinary rectangular room right now, take a look at a corner on the floor. Visualize the *x*-axis and *y*-axis starting at that corner and running along the edges of the floor. Say the *x*-axis runs along the north-south edge of the floor, and the *y*-axis runs along the east-west edge of the floor (you don't need a compass; decide for yourself what is north, west, east, and south). Now, in that same corner where the *x*-axis and *y*-axis started, the vertical line running up from the floor to the ceiling is the *z*-axis; i.e. the *z*-axis is that seam where the "north" wall and the "west" wall meet.

Essentially, up to now, we have been restricted to drawing graphs on the floor, the *xy*-plane. With the addition of the *z*-variable, we can now rise up off the floor into the third dimension. **Don't worry! This is not a course about trying to draw three-dimensional graphs.** But, it might help to try to visualize what we are dealing with here.

Just as we do for lines, we can graph a plane by plotting the intercepts. Since we are dealing with three variables, *x*, *y*, *z*, set two of them equal to 0 and sub in to the plane equation to compute the remaining variable's intercept.

The table of values for $x+2y+3z=6$ would be:

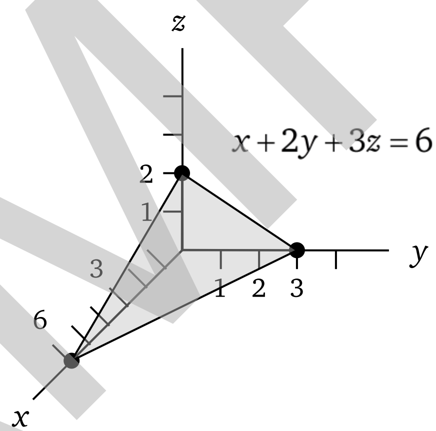
x	y	z
0	0	2
0	3	0
6	0	0

We could now plot these three points and connect the dots to form a triangle. That triangle becomes the base we can rest the entire plane on. Again, look at that corner of the

[*] By the way, a linear equation with 4 variables or more is called a **hyperplane**. This is impossible for the ordinary person to visualize since we are dealing with four dimensions or more in space.

floor where you are visualizing the three axes. The point (6, 0, 0) in our table above tells us to go 6 units along the x-axis and plot a point there (let's say we go 6 inches along our north-south edge); (0, 3, 0) tells us to plot a point 3 units along the y-axis (3 inches along our east-west line); (0, 0 , 2) plots a point 2 units up the z-axis (2 inches up the seam where the "north" and "west" walls meet. If you want, pull out a tape measure and actually try marking those points on the floor and walls (if you don't have a life, I mean). If you were to connect those three dots with some string, you have formed the triangular base that supports the plane. Note, the plane would be making an angle with the floor and walls; it is not parallel to any of them.

Below is how we would attempt to depict this on paper. Note that we only draw the triangle connecting the three intercepts, but it is understood the plane is extending infinitely in all directions from this triangular base it rests upon. Understand we are trying to show three dimensions on two-dimensional paper, so always try to hold on to the image of the walls and floor to properly see this.



> **Let me stress, this is not a course about drawing graphs in three-dimensional space. I am merely doing this as an exercise, so that you might grasp visually what we are dealing with. It is unlikely you will have to draw a graph like this on your exam (it has happened once or twice though, so never say never).**

6. Solve the system of equations below using the elimination method, and interpret the solution geometrically.

$$\begin{cases} \boldsymbol{x+2y+3z=6} \\ \boldsymbol{2x+3y+5z=9} \\ \boldsymbol{5x+2y-z=-2} \end{cases}$$

SOLUTION

There is a two-stage process to the elimination method when three equations are involved. It helps to keep track of things if we number the original equations (1), (2) and (3).

<u>Stage 1</u>: Select a pair of equations and eliminate whichever variable strikes your fancy to create an equation that has only two variables. Number that new equation (4). Then, select a second pair of equations and eliminate the *same* variable. This is a must! If you eliminated x in the first pair, you must eliminate x in the second pair. Number that new equation (5).

<u>Stage 2</u>: Now equations (4) and (5) form a system of equations with two variables. Solve that system by elimination. Once you have solved those two variables, you can sub them into any one of (1), (2) or (3) to solve the remaining variable.

Number the original equations:

$$\begin{cases} (1) \quad x+2y+3z=6 \\ (2) \quad 2x+3y+5z=9 \\ (3) \quad 5x+2y-z=-2 \end{cases}$$

I like that "$-z$" term in equation (3), so I will exploit it to eliminate the "z" terms in my pairs. (Another good choice would be to exploit the "x" term in equation (1) to eliminate the "x" terms in the pairs.)

My first pair will be equations (1) and (3):

$$\begin{array}{lllll} (1) & x+2y+3z=6 & \rightarrow \text{ leave it alone } \rightarrow & x+2y+\cancel{3z} = & 6 \\ (3) & 5x+2y-z=-2 & \rightarrow \text{ multiply by 3 } \rightarrow & \underline{15x+6y-\cancel{3z} =} & \underline{-6} \\ & & \text{Add the columns } \rightarrow & 16x+8y \qquad = & 0 \\ & & \textbf{Equation (4)} & \boldsymbol{16x+8y=0} & \end{array}$$

My second pair will be equations (2) and (3):

$$\begin{array}{lllll}
(2) & 2x+3y+5z=9 & \rightarrow \text{ leave it alone } \rightarrow & 2x+3y+\cancel{5z} = & 9 \\
(3) & 5x+2y-z=-2 & \rightarrow \text{ multiply by 5 } \rightarrow & \underline{25x+10y-\cancel{5z}=} & \underline{-10} \\
 & & \text{Add the columns } \rightarrow & 27x+13y \quad = & -1 \\
 & & \textbf{Equation (5)} & \mathbf{27x+13y=-1} &
\end{array}$$

Equations (4) and (5) now form a system of two equations with two variables:

$$\begin{cases} (4) & 16x+8y=0 \\ (5) & 27x+13y=-1 \end{cases}$$

Here's a good idea: Divide equation (4) by 8 to make the coefficients smaller and easier to work with. (Note: 0 ÷ 8 = 0.)

$$\begin{array}{llll}
(4) & 16x+8y=0 & \rightarrow \text{ divide by 8 } \rightarrow & 2x+y=0 \\
(5) & 27x+13y=-1 & \rightarrow \text{ leave it alone } \rightarrow & 27x+13y=-1
\end{array}$$

Now I will eliminate *y* from this system:

$$\begin{array}{lll}
2x+y=0 & \rightarrow \text{ multiply by } -13 \rightarrow & -26x-\cancel{13y}=0 \\
27x+13y=-1 & \rightarrow \text{ leave it alone } \rightarrow & \underline{27x+\cancel{13y}=-1} \\
 & \text{Add the columns } \rightarrow & x \quad =-1 \\
 & & \mathbf{x=-1}
\end{array}$$

Sub $\mathbf{x = -1}$ into $2x+y=0$:

$$2(-1)+y=0 \rightarrow -2+y=0 \rightarrow \mathbf{y=2}$$

We have established so far $\mathbf{x = -1, y = 2}$. Sub these into any one of the original three equations to solve z. I will use equation (3) $5x+2y-z=-2$:

$$5(-1)+2(2)-z=-2 \rightarrow -5+4-z=-2 \rightarrow -1-z=-2 \rightarrow -z=-1 \rightarrow \mathbf{z=1}$$

Thus, $\mathbf{x = -1, y = 2, z = 1}$ or **(–1, 2, 1)** is the solution to this system. By the way, don't get confused and say this system has three solutions; this system has <u>one</u> solution. That one solution contains values for all three variables.

If time allows, we can check our answer by confirming (−1, 2, 1) satisfies all three of the original equations in the system. If the check fails in any single one of the equations, we have made a mistake.

Subbing $(-1, 2, 1)$ into equation (1) $x+2y+3z=6$, we get:

$$-1+2(2)+3(1)=6 \ \rightarrow\ -1+4+3=6 \ \rightarrow\ 6=6\ \checkmark$$

Subbing $(-1, 2, 1)$ into equation (2) $2x+3y+5z=9$, we get:

$$2(-1)+3(2)+5(1)=9 \ \rightarrow\ -2+6+5=9 \ \rightarrow\ 9=9\ \checkmark$$

Subbing $(-1, 2, 1)$ into equation (3) $5x+2y-z=-2$, we get:

$$5(-1)+2(2)-1=-2 \ \rightarrow\ -5+4-1=-2 \ \rightarrow\ -2=-2\ \checkmark$$

Thus, all three planes pass through the point $(-1, 2, 1)$.

The solution to this system of equations is $x = -1$, $y = 2$, $z = 1$. Interpreting this solution geometrically, we have discovered a graph of these three planes intersects at the point $(-1, 2, 1)$.

Don't even think about trying to draw a graph of these three planes to visualize them intersecting at this one point. It isn't worth the effort, and your picture is probably going to look like somebody spilled the uncooked spaghetti.

Here is a way to get a grasp of this visually. Look at the "north" wall of your room. That's sort of like plane (1). Now look at the "west" wall of your room. That's sort of like plane (2). Note these two planes intersect along the infinite number of points on the <u>line</u> running up the seam where the two walls meet (that seam in the "northwest" corner running from the floor up to the ceiling). Admittedly, these two walls make a right angle with each other, while the two planes in our system may make some other angle, but who cares? Visualize swinging the two walls using that "northwest" seam as a hinge, like swinging the covers of a textbook. The planes can make any angle you want, but they still intersect along that line running up the seam. Finally, look at the floor. That's sort of like plane (3). Note the floor shares a seam with the "north" wall (infinite points along their line of intersection). The floor also shares a seam with the "west" wall (infinite points along their line of intersection). But, there is only one point where the floor meets both the "north" and "west" walls, and that is that point in the corner of the floor at the "northwest" seam. The three planes have a single <u>point</u> of intersection, just as our three planes meet at the point (–1, 2, 1). **(–1, 2, 1) is sort of like that corner where the floor meets both the north wall and the south wall.**

7. Solve the system of equations below using the elimination method, and interpret the solution geometrically.

$$\begin{cases} \mathbf{2x - y - 4z = 0} \\ \mathbf{x + 2y + 3z = 1} \\ \mathbf{2x + y + 5z = 2} \end{cases}$$

SOLUTION

Number the original equations:

$$\begin{cases} (1) \quad 2x - y - 4z = 0 \\ (2) \quad x + 2y + 3z = 1 \\ (3) \quad 2x + y + 5z = 2 \end{cases}$$

I like that "$-y$" term in equation (1), so I will exploit it to eliminate the "y" terms in my pairs. (Another good choice would be to exploit the "x" term in equation (2) to eliminate the "x" terms in the pairs.)

My first pair will be equations (1) and (2):

$$\begin{array}{llcl} (1) & 2x - y - 4z = 0 & \rightarrow \text{ multiply by 2 } \rightarrow & 4x - \cancel{2y} - 8z = 0 \\ (2) & x + 2y + 3z = 1 & \rightarrow \text{ leave it alone } \rightarrow & \underline{x + \cancel{2y} + 3z = 1} \\ & & \text{Add the columns } \rightarrow & 5x \quad - 5z = 1 \\ & & \textbf{Equation (4)} & \mathbf{5x - 5z = 1} \end{array}$$

My second pair will be equations (1) and (3):

$$\begin{array}{llcl} (1) & 2x - y - 4z = 0 & \rightarrow \text{ leave it alone } \rightarrow & 2x - \cancel{y} - 4z = 0 \\ (3) & 2x + y + 5z = 2 & \rightarrow \text{ leave it alone } \rightarrow & \underline{2x + \cancel{y} + 5z = 2} \\ & & \text{Add the columns } \rightarrow & 4x \quad + z = 2 \\ & & \textbf{Equation (5)} & \mathbf{4x + z = 2} \end{array}$$

We now have a system of two equations with two variables:

$$\begin{cases} (4) \quad 5x - 5z = 1 \\ (5) \quad 4x + z = 2 \end{cases}$$

I will eliminate z from this system:

$$\begin{array}{llcl} (4) & 5x - 5z = 1 & \rightarrow \text{ leave it alone } \rightarrow & 5x - \cancel{5z} = 1 \\ (5) & 4x + z = 2 & \rightarrow \text{ multiply by 5 } \rightarrow & \underline{20x + \cancel{5z} = 10} \\ & & \text{Add the columns } \rightarrow & 25x \quad = 11 \end{array}$$

$$\mathbf{x = \frac{11}{25}}$$

Since x = 11/25 is too annoying to sub into one of equations (4) or (5) to solve z, I will perform elimination again; this time eliminating x:

$$\begin{array}{llcl} (4) & 5x-5z=1 & \to \text{ multiply by } -4 \to & -\cancel{20x}+20z=-4 \\ (5) & 4x+z=2 & \to \text{ multiply by } 5 \to & \underline{\cancel{20x}\quad+5z=10} \\ & & \text{Add the columns } \to & 25z=\;6 \\ & & & \boldsymbol{z=\frac{6}{25}} \end{array}$$

We have established so far **x = 11/25, z = 6/25**. Sub these into any one of the original three equations to solve y. I will use equation (1) $2x-y-4z=0$:

$$2\left(\frac{11}{25}\right)-y-4\left(\frac{6}{25}\right)=0 \;\to\; \frac{22}{25}-y-\frac{24}{25}=0 \;\to\; -y-\frac{2}{25}=0 \;\to\; -y=\frac{2}{25} \;\to\; \boldsymbol{y=-\frac{2}{25}}$$

Thus, **x = 11/25, y = –2/25, z = 6/25** or **(11/25, –2/25, 6/25)** is the solution to this system.

If time allows, we can check our answer by confirming (11/25, –2/25, 6/25) satisfies all three of the original equations in the system. If the check fails in any single one of the equations, we have made a mistake. I will leave you to perform the check yourself.

The solution to this system of equations is $\boldsymbol{x=\frac{11}{25},\; y=-\frac{2}{25},\; z=\frac{6}{25}}$. Interpreting this solution geometrically, we have discovered a graph of these three planes intersects at the point $\boldsymbol{\left(\frac{11}{25},-\frac{2}{25},\frac{6}{25}\right)}$.

8. Solve the system of equations below using the elimination method, and interpret the solution geometrically.

$$\begin{cases} \mathbf{2x - y + 3z = 3} \\ \mathbf{-3x + 2y - z = 8} \end{cases}$$

SOLUTION

Wait a minute! There are only two equations here! All we can do then is eliminate one of the variables. I like that "–*y*" term in equation (1), so I will exploit it to eliminate the "*y*" terms. (Another good choice would be to exploit the "–*z*" term in equation (2) to eliminate the "*z*" terms.)

$$\begin{array}{lllll} (1) & 2x - y + 3z = 3 & \rightarrow \text{ multiply by 2 } \rightarrow & 4x - \cancel{2y} + 6z = 6 \\ (2) & -3x + 2y - z = 8 & \rightarrow \text{ leave it alone } \rightarrow & \underline{-3x + \cancel{2y} - z = 8} \\ & & \text{Add the columns } \rightarrow & x \quad + 5z = 14 \\ & & \textbf{Equation (3)} & \mathbf{x + 5z = 14} \end{array}$$

That's as far as we can go. The solution to this system of two equations is the equation $x + 5z = 14$. Note: this is a linear equation with two variables in it. That means it graphs as a line! This makes perfect sense. The original system was two planes, and we have discovered these planes have a **line of intersection**. Again, just like the "north" wall and the "west" wall intersect along that seam running up the northwest corner of your room, two planes can intersect along an infinite line. **There are infinite points of intersection between these two planes, all of them lying on the line $\mathbf{x + 5z = 14}$.**

Just as we did in question 5 above, whenever we have infinite solutions to a system of equations, we will introduce a parameter. The easiest thing here is to make z the parameter (but you could make x the parameter if you prefer).

I will let z = t, a parameter. Subbing $z = t$ into $x + 5z = 14$, we get:

$$x + 5t = 14 \rightarrow \mathbf{x = 14 - 5t}$$

We have established so far $\mathbf{x = 14 - 5t,\ z = t}$. Sub these into either one of the original two equations to solve y. I will use equation (1) $2x - y + 3z = 3$:

$$2(14 - 5t) - y + 3(t) = 3 \rightarrow 28 - 10t - y + 3t = 3 \rightarrow 28 - 7t - y = 3$$

Move everything over to the right side of the equation except the "–*y*" term:

$$-y = 3 - 28 + 7t \rightarrow -y = -25 + 7t \rightarrow \text{multiply both sides by } -1 \rightarrow \mathbf{y = 25 - 7t}$$

Thus, $\boldsymbol{x = 14 - 5t, y = 25 - 7t, z = t}$ or $\boldsymbol{(14 - 5t, 25 - 7t, t)}$ is the solution to this system. We have given people a "recipe" to generate the infinite number of points that satisfy this system of equations. By selecting different values of the parameter t, we generate different solutions. For example, if $t = 0$, we get the solution (14, 25, 0); if $t = 1$, we get the solution (9, 18, 1); if $t = 2$, we get (4, 11, 2); etc.

That is the beauty of using parameters to describe infinite solutions: we get an easy recipe to generate all the solutions. We can let t be any real number. (The parameter t doesn't have to be just counting numbers like 0, 1, 2, …; we can let t be 1/3, $\sqrt{5}$, –4.72, whatever, and they all generate solutions to the system.)

Let's prove $(14 - 5t, 25 - 7t, t)$ is the solution to the system by showing it satisfies both of the equations.

Subbing $(14 - 5t, 25 - 7t, t)$ into equation (1) $2x - y + 3z = 3$, we get:

$$2(14-5t)-(25-7t)+3(t)=3 \;\rightarrow\; 28-10t-25+7t+3t=3 \;\rightarrow\; 3=3\ ✓$$

Note, the t terms cancel out.

Subbing $(14 - 5t, 25 - 7t, t)$ into equation (2) $-3x + 2y - z = 8$, we get:

$$-3(14-5t)+2(25-7t)-(t)=8 \;\rightarrow\; -42+15t+50-14t-\ t\ =8 \;\rightarrow\; 8=8\ ✓$$

Note, the t terms cancel out.

The solution to this system of equations is $\boldsymbol{x = 14 - 5t, y = 25 - 7t, z = t}$ where $\boldsymbol{t}$ is any real number.* Interpreting this solution geometrically, we have discovered a graph of these two planes has a <u>line of intersection</u>. All the points in the form $\boldsymbol{(14 - 5t, 25 - 7t, t)}$ are on this line.

* Be sure to point out that t is any real number. It is generally taken for granted that t, being a parameter, is any real number, but some profs will deduct marks if you don't specifically say this in your answer.

LECTURE PROBLEMS

For your convenience, here are the 8 questions I used as examples in this lesson. Do not make any marks or notes on these questions below. You want to keep these questions untouched, so that you can look back at them without any hints. Instead, make any necessary notes, highlights, etc. in the lecture in Lesson 1 above.

For questions 1 to 8 below, solve the system using the elimination method, and interpret the solution geometrically.

1. $\begin{cases} 2x+3y=6 \\ 5x+2y=-7 \end{cases}$

2. $\begin{cases} \dfrac{x}{4}+y=1 \\ \dfrac{x}{6}-\dfrac{5y}{3}=3 \end{cases}$

3. $\begin{cases} 3x+4y=5 \\ 5x-2y=12 \end{cases}$

4. $\begin{cases} 2x+3y=6 \\ 4x+6y=6 \end{cases}$

5. $\begin{cases} x-4y=4 \\ -2x+8y=-8 \end{cases}$

6. $\begin{cases} x+2y+3z=6 \\ 2x+3y+5z=9 \\ 5x+2y-z=-2 \end{cases}$

7. $\begin{cases} 2x-y-4z=0 \\ x+2y+3z=1 \\ 2x+y+5z=2 \end{cases}$

8. $\begin{cases} 2x-y+3z=3 \\ -3x+2y-z=8 \end{cases}$

Lesson 2: Cost & Revenue

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

1. Find the equations of the following lines:

(a) The line with a slope of 5/3 and a y-intercept of –2.

(b) The line passing through the points (2, –3) and (6, 7).

(c) The line passing through the point (1, 1) and perpendicular to the line $2x-7y=5$.

(d) The line passing through the point (1, –1) and parallel to the line $y=4x+6$.

2. The cost C (in thousands of dollars) of a company that produces q widgets is given by $C=12q+40$.

(a) What is the cost of producing 50 widgets?

(b) How many widgets would cause a cost of $124,000?

3. The table below shows the number of inhabitants n (in thousands) in three cities and the amount of garbage produced each week g (in hundreds of metric tons).

n	20	25	40
g	17	35	89

(a) Does this data show a linear trend?

(b) Use this data to state g as a function of n.

4. In 1990, a company's sales were 20 million dollars. In 2000, they were 27 million dollars. Assuming the trend is linear, predict the sales in 2003.

5. A theatre has a fixed cost of $3,000 per day and a variable cost of $2 per customer. The admission fee is $6 per customer.

(a) Find the cost and revenue functions. How many customers are needed to break even?

(b) Find the profit function and illustrate the break even point calculated in part (a) by sketching a graph of the profit function.

(c) What is the marginal cost, marginal revenue and marginal profit?

Equations of Lines

$y = mx + b$ is the equation of a straight line.

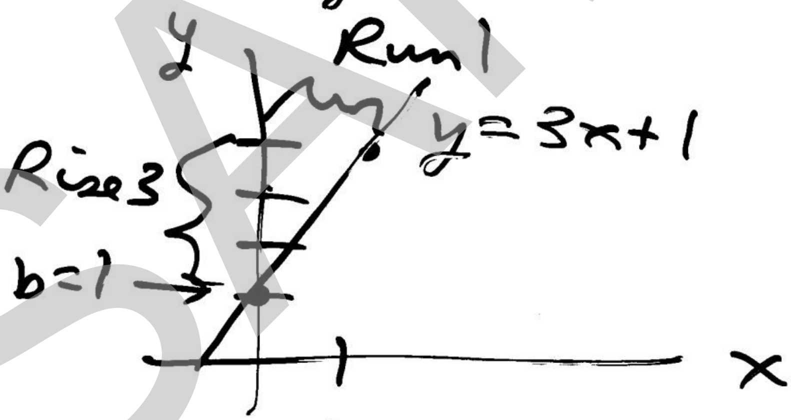
m = the slope of the line $\left(= \frac{\text{Rise}}{\text{Run}}\right)$

b = the y-intercept (where the line crosses the y-axis)

eg. $y = 3x + 1 \rightarrow m = 3 = \frac{3}{1}$

$b = 1$

To graph this line
plot "b" on the y-axis
then count $\frac{\text{Rise}}{\text{Run}}$ from b
to get a 2nd point.



$m = \frac{\text{Rise}}{\text{Run}} = \frac{3}{1}$

Up 3 over 1

We can, of course, graph any line by simply plotting 2 points.

To get the equation of a line we need:
A point on the line (x_0, y_0)
and the slope of the line "m".
We can then use the point-slope formula to get the equation:

$$\boxed{y - y_0 = m(x - x_0)}$$ Memorize

We then can rearrange this formula into one of these forms:

1. Slope-intercept form

$$\boxed{y = mx + b}$$ (b is the y-intercept)

2. Standard form

$$\boxed{ax + by = c}$$ *(This form is rarely used in this course.)*

Lecture Problems:

1.(a) Given: $m = \frac{5}{3}$ and $b = -2$

This is perfect for $y = mx + b$ form!

Answer: $\boxed{y = \frac{5}{3}x - 2}$

To convert to standard form, simply move the x term to the LHS

$$\boxed{-\frac{5}{3}x + y = -2}$$

Traditionally, we remove fraction when using standard form, so multiply every term by 3:

$$\cancel{3}\left(-\frac{5}{\cancel{3}}x\right) + 3(y) = 3(-2) \rightarrow \boxed{-5x + 3y = -6}$$

1.(b) Given: points $(\underset{x_1}{2}, \underset{y_1}{-3})$ and $(\underset{x_2}{6}, \underset{y_2}{7})$

Get the slope: $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

$$m = \frac{7 - (-3)}{6 - 2} = \frac{10}{4} = \frac{5}{2}$$

Use point-slope formula: $y - y_1 = m(x - x_1)$

$$y - (-3) = \frac{5}{2}(x - 2) \rightarrow y + 3 = \frac{5}{2}x - 5$$

$\boxed{y = \frac{5}{2}x - 8}$ OR $-\frac{5}{2}x + y = -8 \rightarrow \boxed{-5x + 2y = -16}$

1.(c) Note: If 2 lines (L_1 and L_2) are perpendicular, then their slopes (m_1 and m_2) are negative recipricals.

ie. $\boxed{\text{If } L_1 \perp L_2 \text{, then } m_1 = \frac{-1}{m_2}}$

Given: point $(\underset{x_1}{1}, \underset{y_1}{1})$ and the line $2x-7y=5$

convert the line to $y=mx+b$ form in order to read off m:

$2x-7y=5 \rightarrow -7y=-2x+5 \rightarrow$ divide by -7

$$\frac{\cancel{-7}y}{\cancel{-7}} = \frac{-2x}{-7} + \frac{5}{-7} \rightarrow \underset{m}{y = \frac{2}{7}} \underset{x}{x} \underset{+b}{- \frac{5}{7}}$$

This line has $m=\frac{2}{7}$; our line is perpendicular, so its slope is $m=-\frac{7}{2}$ (the negative reciprical)

Thus: $(x_1, y_1)=(1,1)$; $m=-\frac{7}{2}$

$y-y_1=m(x-x_1) \rightarrow y-1=-\frac{7}{2}(x-1)$

$y-1=-\frac{7}{2}x+\frac{7}{2}+1$ (Note: $\frac{7}{2}+1=\frac{7+2}{2}=\frac{9}{2}$)

$\boxed{y=-\frac{7}{2}x+\frac{9}{2}}$ OR $\frac{7}{2}x+y=\frac{9}{2} \rightarrow \boxed{7x+2y=9}$

1.(d) Given: $(\underset{x_1}{1}, \underset{y_1}{-1})$ and parallel line $y = 4x + 6$

If 2 lines are parallel they have the same slope. ie. $L_1 \parallel L_2$ then $m_1 = m_2$

Given $y = \underset{m}{4}\underset{x}{x} + \underset{b}{6} \rightarrow m = 4$

$(x_1, y_1) = (1, -1)$

$y - y_1 = m(x - x_1) \rightarrow y - (-1) = 4(x - 1)$

$y + 1 = 4x - 4 \rightarrow \boxed{y = 4x - 5}$

OR $\boxed{-4x + y = -5}$

2.(a) $C = 12q + 40$

$q = 50 \rightarrow C = \underbrace{12(50)}_{600} + 40 = 640$

$\boxed{\text{The cost is 640 thousand dollars.}}$

2.(b) $C = 124$ (C is in thousands!)

$124 = 12q + 40$

$84 = 12q \rightarrow \frac{\cancel{12}q}{\cancel{12}} = \frac{84}{12} = 7$

$\boxed{\text{7 widgets would cost \$124,000.}}$

3.

n	20	25	40
g	17	35	89

3.(a) We have been given three points

$(20, 17)$ $(25, 35)$ $(40, 89)$
x_1 y_1 x_2 y_2 x_3 y_3

If this is linear, the slope of any 2 of these points should be the same. Use the first point as an "anchor": Find "m" for that first point together with all the other point. m must be the same every time to be linear.

1st & 2nd point: $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{35 - 17}{25 - 20} = \dfrac{18}{5}$

1st & 3rd: $m = \dfrac{y_3 - y_1}{x_3 - x_1} = \dfrac{89 - 17}{40 - 20} = \dfrac{72 \div 4}{20 \div 4} = \dfrac{18}{5}$ ✓

Yes, this data is linear (all 3 points fall on the same line).

3.(b) g as a function of n

<u>Note</u>: $y = mx + b$ states y as a function of x (y <u>depends</u> on x)

Since y is isolated, we are giving y as a function of x.

eg. $y = 2x + 3$

In problem 2, $C = 12q + 40$

C is a function of q.

g is sort of our y

n is sort of our x

$(x_1, y_1) = (\underset{n}{20}, \underset{g}{17})$; $m = \frac{18}{5}$

$y - y_1 = m(x - x_1) \rightarrow y - 17 = \frac{18}{5}(x - 20)$

$y - 17 = \frac{18}{5}x - \frac{\cancel{360}^{72}}{\cancel{5}}$

$\underset{g}{y} = \frac{18}{5}\underset{n}{x} - 55$

$\therefore \boxed{g = \frac{18}{5}n - 55}$ Don't change to standard form. This gives g as a function of n

4. Predict the sales → "y"

$y = mx + b$ is perfect for predicting y

Let x = the year, y = the sales (in millions \$)

Given: $(\underset{x_1}{1990}, \underset{y_1}{20})$ and $(\underset{x_2}{2000}, \underset{y_2}{27})$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{27 - 20}{2000 - 1990} = \frac{7}{10} = m$$

$$y - y_1 = m(x - x_1) \rightarrow y - 20 = \frac{7}{10}(x - 1990)$$

$$y - 20 = \frac{7}{10}x - \frac{7(199\cancel{0})}{1\cancel{0}}$$

Wait a minute, it would have been easier to use the point $(\underset{x_1}{2000}, \underset{y_1}{27})$

$$y - 27 = \frac{7}{10}(x - 2000) \rightarrow y - 27 = \frac{7}{10}x - \underbrace{\frac{1400\cancel{0}}{1\cancel{0}}}_{-1400+27}$$

$y = \frac{7}{10}x - 1373$ predicts the sales for any year "x"

$$x = 2003 \rightarrow y = \frac{7}{10}(2003) - 1373$$

STOP HERE if time is an issue

Sales will be $\frac{7}{10}(2003) - 1373$ million dollars

(ie $\frac{14021}{10} - 1373 = \underset{1373}{1402.1} - 1373 = 29.1$ million dollars

In word problems they will often give you a fixed # and a variable rate

eg. The cost of manufacturing a product has a fixed cost of \$500 and has a variable cost of \$10 per unit produced.

→ fixed cost = b, the y-intercept
variable cost = m, the slope
x = # of units produced.
cost per unit

Let C = the cost (\$)

$$y = mx + b$$
$$C = 10x + 500$$

Note: The Fixed cost is the cost of producing x = 0 units!

Cost and Revenue Problems

C = Cost = amount of money ($) we spend in producing a product

R = Revenue = amount of money ($) we get for selling the product

Breakeven Point

Revenue = Cost

Profit = Revenue − Cost

5.(a) fixed cost of $3000 → b
variable cost of $2 per customer → m

x = # of customers

$y = mx + b$

$$\boxed{C = 2x + 3000}$$

OR Function Notation

$$C(x) = 2x + 3000$$

Revenue: $6 per customer
(variable revenue → m = 6)
No fixed revenue → b = 0

$y = mx + \cancel{b}$

$\boxed{R = 6x}$ OR $R(x) = 6x$

Breakeven? Revenue = Cost

$R = 6x$

$C = 2x + 3000$ → Set $R = C$

$6x = 2x + 3000$

$4x = 3000$

$x = \frac{3000}{4} = \frac{1500}{2} = 750$

$\boxed{\text{We need 750 customers to break even.}}$

5.(b) Profit = Revenue − Cost

$P = R - C$

$P = 6x - (2x + 3000)$ Brackets!

$\boxed{P = 4x - 3000}$ is the profit function

Plot 2 points

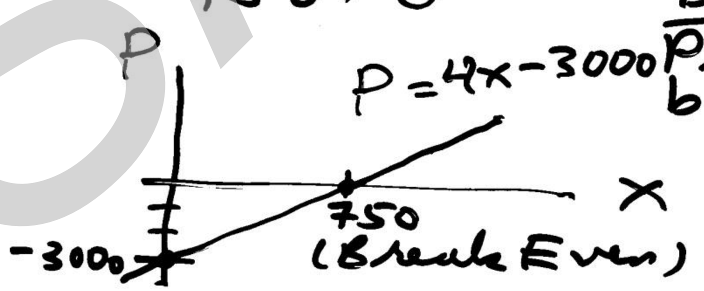
→ Plot intercept → $b = -3000$

→ Plot Breakeven point → $x = 750$

x	P
750	0

→ $P = 0$ b/c it is Breakeven

Profit = 0 when you breakeven



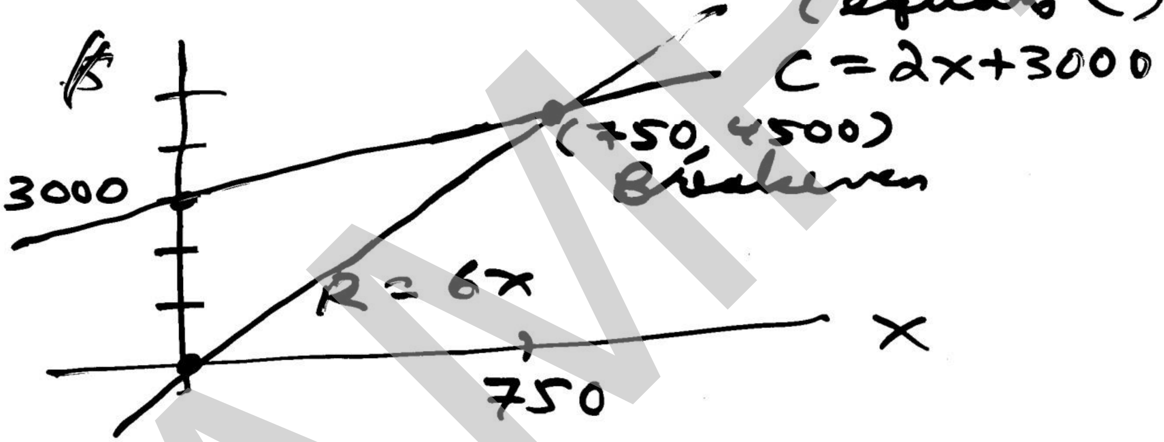
Let's Graph the Cost & Revenue Lines and illustrate the Breakeven point. Cost = Revenue when the 2 lines cross

$C = 2x + 3000$, $R = 6x$

$x = 750$ is breakeven

C: $b = 3000$; $x = 750$, $C = 4500$

R: $b = 0$; $x = 750$, $R = 4500$ ✓ (equals C)



5.(c) Marginal Cost (MC) is the rate of change of the cost.

MC is the cost of producing the "next unit".

MC is the slope of the Cost line

Similarly Marginal Revenue (MR) is the expected revenue for the "next unit" sold

MR is the slope of the Revenue line

Marginal Profit (MP or $M\pi$) is the slope of the profit line (expected profit from selling the next unit.)

We have
$$C = 2x + 3000 \rightarrow m = 2$$
$$R = 6x \rightarrow m = 6$$
$$\pi = 4x - 3000 \rightarrow m = 4$$

Marginal Cost = \$2 per customer
(marginal values are \$ per unit)

Marginal Revenue = \$6 per customer

Marginal Profit = \$4 per customer

Homework:

- Study the lesson thoroughly until you can do all of **questions 1 to 5** on page 20 from start to finish without any assistance.
- Do all of the **Practise Problems** below (solutions are on page 36).

Practise Problems:

1. Producing x cars costs $C(x) = 10x + 150$ thousand dollars. Each car is sold for 20 thousand dollars.

(a) Find the revenue function $R(x)$.

(b) How many cars must be sold to break even?

(c) What is the profit if 50 cars are sold?

(d) How many must be sold for a profit of $400,000?

(e) Sketch $C(x)$ and $R(x)$ on the same graph.

2. A firm producing the Latest Craze kid's doll finds the total cost of producing and selling x dolls is given by $C(x) = 20x + 3600$. They will charge $60 per doll.

(a) How many dolls must be sold to break even?

(b) What is the profit if 100 dolls are sold?

(c) How many must be sold to produce a profit of $10,000?

(d) What is the average cost per doll if 50 are produced?

3. A factory produces radios. The cost of producing x radios is $C(x) = 13x + 2400$ dollars, and they are sold for $25 each.

(a) What is the marginal cost?

(b) Find the revenue (income) function $R(x)$. What is the break-even point?

(c) Find the profit function $P(x)$. What is the marginal profit?

4. A small company produces doohickeys. It costs \$450 to produce 5 doohickeys, and for each additional doohickey, the total cost increases \$30.

(a) Find a linear function $C(x)$ for producing x doohickeys.

(b) If the selling price is \$45 per doohickey, find the revenue function $R(x)$.

(c) Find the break-even quantity if all doohickeys produced are sold.

5. A small firm produces paperweights. The first paperweight costs \$25 to produce, and each additional paperweight costs \$5 more.

(a) Find the linear cost function $C(x)$ for producing x paperweights.

(b) If the price is \$9 per paperweight, find the revenue function.

(c) Find the **break-even** quantity, if all paperweights produced are sold.

(d) How many paperweights must be sold to make a profit of \$1000?

1. (a) $\boxed{R(x)=20x \text{ thousand dollars.}}$

1. (b) Break-even occurs when $R(x)=C(x)$:

$$20x=10x+150 \to 10x=150 \to \boxed{x=\frac{150}{10}=15}$$

They must sell 15 cars to break even.

1. (c) *Profit = Revenue – Cost* $\to$ $P(x)=R(x)-C(x)$:

$$P(x)=20x-(10x+150) \to \boxed{P(x)=10x-150}$$

$$P(50)=10(50)-150=500-150=\boxed{350}$$

The profit if 50 cars are sold is $350,000.

1. (d) Remember: we are using <u>thousands</u> of dollars for our units. Set $P(x)$ equal to 400 thousand dollars <u>not</u> 400,000!

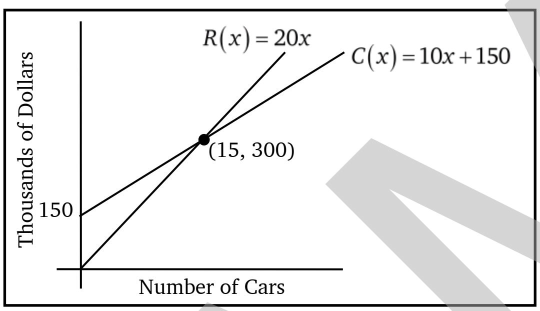
$$P(x)=10x-150=400 \to 10x=550 \to \boxed{x=\frac{550}{10}=55}$$

To make a profit of $400,000 sell 55 cars.

1. (e) For $C(x)=10x+150$, clearly the y-intercept is 150. For a second point, the best choice is the break-even point found in (b) above:

When $x=15: C(15)=10(15)+150=300$ $\to$ Plot (15, 300).

For $R(x)=20x$, clearly the y-intercept is 0 and it also passes through the point (15, 300), since the **Cost and Revenue graphs always intersect at the break-even point.** Thus:



DON'T FORGET: LABEL YOUR AXES WHEN DRAWING GRAPHS.

2. (a) Clearly: $R(x)=60x$ $\to$ Set $R(x)=C(x)$:

$$60x=20x+3600 \to 40x=3600 \to \boxed{x=\frac{360\cancel{0}}{4\cancel{0}}=90}$$

They must sell 90 dolls to break even.

2. (b) *Profit = Revenue – Cost* $\to$ $P(x)=R(x)-C(x)$:

$$P(x)=60x-(20x+3600) \to \boxed{P(x)=40x-3600}$$

$$P(100)=40(100)-3600=4000-3600=\boxed{400}$$

If 100 dolls are sold, the profit is $400.

2. (c)

$$P(x)=40x-3600=10{,}000 \to 40x=13{,}600 \to \boxed{x=\frac{13{,}60\cancel{0}}{4\cancel{0}}=340}$$

To produce a profit of $10,000, they must sell 340 dolls.

2. (d) $\text{average cost}=\dfrac{\text{total cost of production}}{\text{number of items produced}}=\dfrac{C(x)}{x}$

$$\boxed{x=50}: C(50)=20(50)+3600=1000+3600=\boxed{4600}$$

It costs $4600 to produce 50 dolls: $\text{average cost}=\dfrac{460\cancel{0}}{5\cancel{0}}=\boxed{92}$

The average cost if 50 are produced is $92 per doll.

3. (a) Recall: marginal cost is the *slope* of the cost line. By $y=mx+b$ form: $C(x)=\underset{m}{13}x+\underset{b}{2400}$, we see the slope, m=13.

The marginal cost is $13 per radio.

3. (b) The radios sell for $25 each: $R(x)=25x$.

Break-even occurs when $R(x)=C(x)$:

$$25x=13x+2400 \to 12x=2400 \to \boxed{x=\frac{2400}{12}=200}$$

They must sell 200 radios to break even.

3. (c) *Profit = Revenue – Cost* $\to$ $P(x)=R(x)-C(x)$:

$$P(x)=25x-(13x+2400) \to \boxed{P(x)=\underset{m}{12}x-2400}$$

The marginal profit is the *slope*, m = 12.

The profit function is $P(x)=12x-2400$ **and the marginal profit is $12 per radio.**

4. (a) It costs $450 to produce 5 doohickeys: When x=5, C=450. i.e. We have been given the **point (5, 450)**.
For each additional doohickey, the total cost increases $30. This gives us the *variable cost* $\to$ the **slope m=30**.
Use the point-slope formula:

$$\underset{450}{y-y_0}=\underset{30}{m}(x-\underset{5}{x_0}) \to y-450=30(x-5)$$

$$y-450=30x-150 \to \boxed{y=30x+300}$$

But y is the cost, $C(x)$, therefore:

The cost function is $C(x)=30x+300$**.**

4. (b) The selling price is $45 per doohickey: $\boxed{R(x)=45x}$.

4. (c) Break-even when $R(x)=C(x)$:

$$45x=30x+300 \to 15x=300 \to \boxed{x=\frac{300}{15}=20}$$

They must sell 20 doohickeys to break even.

5. (a) The first paperweight costs $25 to produce: When x=1, C=25. i.e. We have been given the **point (1, 25)**.
Each additional paperweight costs $5 more. This gives us the *variable cost* $\to$ the **slope m=5**.
Use the point-slope formula:

$$\underset{25}{y-y_0}=\underset{5}{m}(x-\underset{1}{x_0}) \to y-25=5(x-1)$$

$$y-25=5x-5 \to \boxed{y=5x+20}$$

But y is the cost, $C(x)$, therefore:

The cost function is $C(x)=5x+20$**.**

5. (b) The price is $9 per paperweight: $\boxed{R(x)=9x}$

5. (c) Break-even when $R(x)=C(x)$:

$$9x=5x+20 \to 4x=20 \to \boxed{x=\frac{20}{4}=5}$$

They must sell 5 paperweights to break even.

5. (d) *Profit = Revenue – Cost* $\to$ $P(x)=R(x)-C(x)$:

$$P(x)=9x-(5x+20) \to \boxed{P(x)=4x-20}$$

Set $P(x)$ equal to $1000 and solve for x:

$$P(x)=4x-20=1000 \to 4x=1020 \to \boxed{x=\frac{1020}{4}=255}$$

They must sell 255 paperweights to make a $1000 profit.

Four ways Grant can help you:

- ## Grant's Study Books

    - **Basic Statistics 1 (Stat 1000)**
    - **Basic Statistics 2 (Stat 2000)**
    - **Linear Algebra and Vector Geometry (Math 1300)**
    - **Matrices for Management (Math 1310)**
    - **Intro Calculus (Math 1500 or Math 1510)**
    - **Calculus for Management (Math 1520)**
    - **Calculus 2 (Math 1700 or 1710)**

  All these books are available at **UMSU Digital Copy Centre**, room 118 University Centre, University of Manitoba. **Grant's books can be purchased there all year round. You can also order a book from Grant directly.** Please allow one business day because the books are made-to-order.

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Lesson 3: Row-Reduction and Linear Systems

The Rank of a Matrix:

- ☑ The rank of a matrix equals the number of leading 1's it would have in its row-reduced echelon form.
- ☑ If a system is consistent (one or infinite solutions), the rank of the coefficient matrix is equal to the rank of the augmented matrix.
- ☑ If a system is inconsistent, the rank of the coefficient matrix is less than the rank of the augmented matrix. (The augmented matrix will have a rank that is one higher than the coefficient matrix.)
- ☑ The rank of the coefficient matrix could never be more than the rank of the augmented matrix.

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

1. Suppose that the following matrices are the row echelon form of the augmented matrix of a system of linear equations. For each matrix answer the following questions:

(i) How many equations and how many variables were in the original system?

(ii) What is the rank of the coefficient matrix and the augmented matrix?

(iii) How many parameters are in the solution?

(iv) List the solution(s), if possible.

(a) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right)$

(b) $\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & -2 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$

(c) $\left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 & -5 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$

2. Consider the system
$$\begin{array}{rcrcrcl} 2x & + & 3y & + & z & = & a \\ x & & & + & z & = & b. \\ & & y & - & 2z & = & c \end{array}$$

Suppose (1, 2, –1) is a solution to this system, find a, b and c.

3. Solve the following systems of equations using Gauss-Jordan elimination.

(a)
$$\begin{array}{rcrcrcrcrcr} 2x_1 & + & 2x_2 & - & x_3 & & & + & x_5 & = & 2 \\ -x_1 & - & x_2 & + & 2x_3 & - & 3x_4 & + & x_5 & = & 5 \\ x_1 & + & x_2 & - & 2x_3 & & & - & x_5 & = & -2 \\ & & & & x_3 & + & x_4 & + & x_5 & = & 1 \end{array}$$

(b)
$$\begin{array}{rcrcrcr} 3x & + & 7y & + & 2z & = & 9 \\ 2x & + & 4y & + & 2z & = & 4 \\ x & + & 3y & - & z & = & 4 \end{array}$$

(c)
$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & + & x_3 & + & x_4 & = & 1 \\ 2x_1 & + & 3x_2 & + & 3x_3 & & & = & 1 \\ -x_1 & - & 2x_2 & - & 2x_3 & + & x_4 & = & 0 \\ & - & x_2 & - & x_3 & + & 2x_4 & = & 1 \end{array}$$

(d)
$$\begin{array}{rcrcrcr} 2x & + & y & + & z & = & 2 \\ & & y & - & z & = & -1 \\ x & & & + & z & = & 1 \end{array}$$

4. Solve the system of equations

$$\begin{array}{rcrcrcr} & - & y & + & z & = & 3 \\ x & - & y & - & z & = & 0 \\ -x & & & - & z & = & -3 \end{array}$$

using Gaussian elimination and back substitution.

5. Solve the two systems of equations below simultaneously:

$$\begin{array}{rcrcrcl} x & + & 6y & + & 3z & = & 34 \\ x & + & 6y & + & 2z & = & 30 \\ & & 2y & + & 2z & = & 14 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcl} x & + & 6y & + & 3z & = & 30 \\ x & + & 6y & + & 2z & = & 24 \\ & & 2y & + & 2z & = & 16 \end{array}$$

6. Given the system of equations

$$\begin{array}{rcrcrcl} x_1 & - & x_2 & + & 2x_3 & = & 0 \\ & & x_2 & - & x_3 & = & k, \\ -x_1 & + & 2x_2 & - & 3x_3 & = & 1 \end{array}$$

find, if possible, the value of k if

(a) the system has infinite solutions.

(b) the system has a unique solution.

(c) the system has no solution.

7. Given the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & a & b \end{array}\right),$$

find conditions on real numbers a and b such that:

(a) the system has no solution.

(b) the system has a unique solution.

(c) the system has infinitely many solutions.

8. A linear system of equations has been row-reduced into this augmented matrix (it is not necessarily in RREF)

$$\left(\begin{array}{ccc|c} 1 & 0 & a+1 & 7 \\ 0 & 1 & -5 & 6 \\ 0 & 0 & a^2-4a & a-4 \end{array}\right),$$

find all real numbers a such that:

(a) the system has infinitely many solutions.

(b) the system has no solution.

(c) the system has a unique solution.

9. Anne, Betty and Carol went to their local produce store to purchase some fruit. Anne bought one pound of apples and two pounds of bananas and paid \$1.85. Betty bought two pounds of apples and one pound of grapes and paid \$3.65. Carol bought one pound of bananas and two pounds of grapes and paid \$3.95. Find the price per pound for each of the three fruits.

10. A company owns three types of trucks. These trucks are equipped to haul two different types of machines per load. Truck 1 can haul 2 of machine A and 3 of machine B. Truck 2 can haul 1 of machine A and 2 of machine B. Truck 3 can haul 3 of machine A and 4 of machine B. Assuming each truck is fully loaded, how many trucks of each type should be sent to haul exactly 18 of machine A and 26 of machine B. If there is more than one possible solution provide all possible solutions, keeping in mind that the company can use no more than 4 of any particular type of truck.

11. List <u>all</u> 3×2 row-reduced echelon form matrices.

12. Consider the linear equation with three variables: $ax + by + cz = d \quad (1)$

where a, b, c, and d are any real number but $d \neq 0$.

Then, the associated homogeneous equation would be: $ax + by + cz = 0 \quad (2)$.

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be two solutions to equation (1), and let (x_0, y_0, z_0) be a solution to equation (2).

(a) Show $(x_1 - x_2, y_1 - y_2, z_1 - z_2)$ is a solution to equation (2).

(b) Show $(x_1 - x_0, y_1 - y_0, z_1 - z_0)$ is a solution to equation (1).

(c) Show (kx_0, ky_0, kz_0) is a solution to equation (2) where k is any real number.

ROW-REDUCED ECHELON FORM (RREF)

A matrix is in row-reduced echelon form (RREF for short) if it satisfies these four conditions.

(Note that rows are horizontal (↔) and columns are vertical (↕).)

1. A row consisting strictly of 0's is called a "Zero Row". If a RREF matrix has any zero rows at all, they must be the last rows in the matrix. There does not have to be any zero rows at all in a RREF matrix
2. As you read from left to right, the first non-zero entry in each row must be a "1". This is called a "leading 1".
3. As you go down the rows, each leading 1 must appear further to the right than all preceding leading 1's.
4. Any column that contains a leading 1 must have "0" for every other entry in that column above and below the "1".

eg. $\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is a RREF matrix

(I have circled the leading 1's so you can see how all four conditions are met.)

Note: Row Echelon Form (REF) is identical to RREF except, in the 4th condition we only require 0's below the leading 1's.

eg $\begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is REF

We will never use REF, we want RREF.

Solving Linear Systems of Equations

Our goal is to express the system as an augmented matrix and then row-reduce it into RREF.

eg. $\begin{aligned} 2x + 3y &= 5 \\ 4x - 7y &= 10 \end{aligned}$ is $\underbrace{\left(\underbrace{\begin{matrix} 2 & 3 \\ 4 & -7 \end{matrix}}_{\text{coefficient matrix}} \middle| \underbrace{\begin{matrix} 5 \\ 10 \end{matrix}}_{\text{constant matrix}}\right)}_{\text{augmented matrix}}$

The Beauty of RREF is that we can read off the solution of a system.

eg. $\begin{array}{cccc} x & y & z & \\ \end{array}$

$\left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \middle| \begin{matrix} 2 \\ 3 \\ 4 \end{matrix}\right) \begin{matrix} R_1: & x = 2 \\ R_2: & y = 3 \\ R_3: & z = 4 \end{matrix}$

A linear system will either be consistent (have one OR more solutions) OR inconsistent (have no solution). Specifically, a consistent system has either a unique solution (one specific answer) OR infinitely many solutions (due to the existence of one OR more parameters → t, s, r, etc.)

A system is inconsistent if and only if at any time in the row-reduction we get a row with strictly 0's in the coefficient matrix but nonzero in the constant matrix → $(0\ 0\ 0 \mid \text{nonzero})$

e.g. $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 0 & 0 & 6 \end{array}\right)$ inconsistent

no solution b/c $0 \neq 6$

Note: $(0\ 0\ 0 \mid 0)$ is Fine → $0 = 0$

$(0\ 0\ 6 \mid 0)$ is Fine

This is not saying $6 = 0$

it is saying $6z = 0$

"6" is a coefficient → $z = 0$ is in the solution

(0 0 0 0 | nonzero) is inconsistent

Anything else is consistent.

If a system is consistent, be sure you have put it into RREF. (!)

Then, circle every column in the coefficient matrix that has a leading 1. If any column in the coefficient matrix lacks a leading 1, then that column's variable is a parameter (free variable → can have any real number value → infinite possibilities)

eg:
$$\begin{array}{c} \begin{array}{ccccc} x_1 & s & x_3 & t & x_5 \end{array} \\ \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{array}\right) \end{array}$$

RREF
x_2 and x_4 columns lack leading 1's
Parameters!
$x_2 = s$, $x_4 = t$
Label their columns

R_1: $x_1 + 2s + 3t = 2$

Isolate x_1: $\boxed{x_1 = 2 - 2s - 3t}$

R_2: $x_3 + t = -3$

Isolate x_3: $\boxed{x_3 = -3 - t}$

R_3: $\boxed{x_5 = 5}$

Shortcut:

$$\begin{array}{cccccc} x_1 & s & x_3 & t & x_5 & \\ \end{array}$$
$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{array}\right)$$

$x_1 =$
$x_2 = s$
$x_3 =$
$x_4 = t$
$x_5 =$

Fill in the parameter first

Then, each other unknown is found by isolating the leading 1 term in each row

R_1

$$\begin{array}{cccccc} x_1 & s & x_3 & t & x_5 & \\ \end{array}$$
$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 0 & 2 \end{array}\right)$$

$x_1 = 2 - 2s - 3t$

R_2 will give us x_3

$$\begin{array}{cccccc} x_1 & s & x_3 & t & x_5 & \\ \end{array}$$
$$\left(\begin{array}{ccccc|c} 0 & 0 & 1 & 1 & 0 & -3 \end{array}\right)$$

$x_3 = -3 - t$

R_3 will give us x_5

$$\left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 1 & 5 \end{array}\right)$$

$x_5 = 5$

$$\boxed{\begin{array}{l} x_1 = 2 - 2s - 3t \\ x_2 = s \\ x_3 = -3 - t \\ x_4 = t \\ x_5 = 5 \end{array}}$$

OR, vector form:

$(2-2s-3t, s, -3-t, t, 5)$
s and t are any real number

eg.
$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & -4 & 6 \\ 0 & 1 & 3 & 5 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$
$x_1 \quad x_2 \quad s \quad t$

RREF

x_3 & x_4 are parameters

$x_1 = 6 - 2s + 4t$
$x_2 = 7 - 3s - 5t$
$x_3 = s$
$x_4 = t$

Vector form:
$(6-2s+4t, 7-3s-5t, s, t)$ where s and t are any real number.

Let's look at the Questions

1. (a)
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right)$$
No parameters → unique solution

(i) 3 equations b/c 3 rows
3 variables (x, y, z) b/c there are 3 columns in coefficient matrix.

(ii) Rank = # of leading 1's in RREF matrix

Rank of coefficient matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is 3. Rank of augmented matrix $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right)$ is still 3

(iii) No parameters

(iv) $x = 3$
$y = 5$ OR $(3, 5, -2)$
$z = -2$

1. (b)

$$\begin{array}{cccccc} & s & & t & & \\ \end{array}$$
$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & -2 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

(i) 3 equations (3 rows)
5 variables (5 columns in coefficient matrix)

(ii) Rank of Coeff. Matrix = 3
(3 leading 1's)
Rank of Aug. Matrix = 3

(iii) 2 parameters (s & t)

(iv) $x_1 = 4 - 2s - 3t$
$x_2 = s$
$x_3 = -3 + 2t$
$x_4 = t$
$x_5 = 0$

OR

$\boxed{(4-2s-3t,\ s,\ -3+2t,\ t,\ 0)}$

1. (c)

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 & -5 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

inconsistent
no solution
b/c $0 \neq 1$

(i) 3 equations (3 rows)
4 variables (4 columns in coeff. matrix)

(ii) Rank of coeff. matrix = 2

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2 leading 1's

Rank of augmented matrix = 3
(Inconsistent systems are where Rank of augmented matrix > Rank of coefficient matrix.)

(iii) There are no parameters b/c there is no solution.

(iv) No solution

How do we achieve RREF?

<u>First</u> get a leading 1 in Top Left Corner $\begin{pmatrix} \square & & \\ & & \\ & & \end{pmatrix}$, then 0-out the column below that 1.

<u>Then</u>, get a leading 1 in R_2 (preferably in C_2, if possible) then 0-out the column above and below that 1.

Proceed in this fashion until your last possible row has a leading 1 and its column has be 0-ed out above & below.

How do we get a <u>leading 1</u>?

1. Perhaps a 1 is already there, all we have to do is switch rows

e.g. $\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 2 & 4 & 7 \\ 0 & 1 & 5 & 7 \end{array}\right)$ $R_2 \leftrightarrow R_3$

<u>Note</u>: Not $R_1 \leftrightarrow R_2$ b/c R_1 is off Limits (Ruin his leading 1

2. If a 1 is not present, any nonzero number can be changed into 1 by multiplying by its <u>reciprical</u>.

$2 \cdot \frac{1}{2} = 1$

$3 \cdot \frac{1}{3} = 1$

$-5 \cdot -\frac{1}{5} = 1$

$\frac{2}{3} \cdot \frac{3}{2} = 1$

$\boxed{\text{"0" cannot change into 1}}$

Simply multiply the row by the appropriate reciprical.

eg.
$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 5 & 10 & -5 \\ 0 & 6 & 7 & 3 \end{array}\right) \quad \begin{array}{l} 5 \to 1 \to \text{multiply by } \frac{1}{5} \\ R_2 \to \frac{1}{5} \cdot R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 6 & 7 & 3 \end{array}\right)$$

eg.
$$\left(\begin{array}{ccc|c} 1 & 3 & 7 & -1 \\ 0 & 4 & 12 & 3 \\ 0 & 6 & 3 & 5 \\ 0 & -2 & 8 & 4 \end{array}\right) \quad R_2 \to \frac{1}{4} \cdot R_2$$

But we will get fractions

We could also consider $R_3 \to \frac{1}{6} \cdot R_3$ But Worse

OR $R_4 \to -\frac{1}{2} \cdot R_4$ Perfect

then $R_4 \leftrightarrow R_2$

$$\left(\begin{array}{ccc|c} 1 & 3 & 7 & -1 \\ 0 & 4 & 12 & 3 \\ 0 & 6 & 3 & 5 \\ 0 & -2 & 8 & 4 \end{array}\right) \quad R_4 \to -\tfrac{1}{2}\cdot R_4 \quad \left(\begin{array}{ccc|c} 1 & 3 & 7 & -1 \\ 0 & 4 & 12 & 3 \\ 0 & 6 & 3 & 5 \\ 0 & 1 & -4 & -2 \end{array}\right)$$

Now $R_4 \leftrightarrow R_2$

$$\left(\begin{array}{ccc|c} 1 & 3 & 7 & -1 \\ 0 & 1 & -4 & -2 \\ 0 & 6 & 3 & 5 \\ 0 & 4 & 12 & 3 \end{array}\right)$$

Leading 1 → Multiply by Reciprocal

How do we 0-out above & below the 1?

Simply Add/Subtract the appropriate multiple of leading 1 Row to the Row where you want a 0.

0-out C_2 above & below leading 1 row

eg.
$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & -2 \\ 0 & 1 & 3 & -5 \\ 0 & -3 & 7 & 3 \\ 0 & 5 & 8 & 4 \end{array}\right) \quad \begin{array}{l} R_1 \to R_1 - 2\cdot R_2 \\ \qquad\quad \text{opposite} \\ R_3 \to R_3 + 3\cdot R_2 \\ \\ R_4 \to R_4 - 5\cdot R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 8 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 16 & -12 \\ 0 & 0 & -7 & 29 \end{array}\right)$$

Shortcuts

Note: Any column that has already been reduced to leading 1 with 0's will never change

Also, trust that your math is 0-ing out the column you are working on. All columns prior to this column will not change.

Back to the Questions

2. If $x = 1$, $y = 2$, $z = -1$ is a solution then it should work in the eqns.

$$\underset{(1)}{2x} + \underset{(2)}{3y} + \underset{(-1)}{z} = a \rightarrow 2 + 6 - 1 = a \quad \boxed{a = 7}$$

$$\underset{(1)}{x} \quad \underset{(-1)}{+z} = b \rightarrow 1 - 1 = b \quad \boxed{b = 0}$$

$$\underset{(2)}{y} - \underset{(-1)}{2z} = c \rightarrow 2 + 2 = c \quad \boxed{c = 4}$$

3. (a)

$$\left(\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 2 \\ -1 & -1 & 2 & -3 & 1 & 5 \\ 1 & 1 & -2 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

Leading 1 in R_1

$R_1 \leftrightarrow R_3$

$$\left(\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & -2 \\ -1 & -1 & 2 & -3 & 1 & 5 \\ 2 & 2 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

0-out C_1 below 1

$R_2 \rightarrow R_2 + 1 \cdot R_1$

$R_3 \rightarrow R_3 - 2 \cdot R_1$

Leading 1 in R_2

$$\left(\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & -2 \\ 0 & 0 & 0 & -3 & 0 & 3 \\ 0 & 0 & 3 & 0 & 3 & 6 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

$R_2 \leftrightarrow R_4$

0-out C_3 above/below 1

$$\left(\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 0 & 3 & 6 \\ 0 & 0 & 0 & -3 & 0 & 3 \end{array}\right)$$

$R_1 \rightarrow R_1 + 2 \cdot R_2$

$R_3 \rightarrow R_3 - 3 \cdot R_2$

Leading 1 in C_4

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -3 & 0 & 3 \\ 0 & 0 & 0 & -3 & 0 & 3 \end{array}\right)$$

$R_3 \rightarrow -\frac{1}{3} \cdot R_3$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -3 & 0 & 3 \end{array}\right)$$

0-out C_4 above/below 1

$R_1 \to R_1 - 2 \cdot R_3$

$R_2 \to R_2 - 1 \cdot R_3$

$R_4 \to R_4 + 3 \cdot R_3$

$$\begin{array}{cccccc} x_1 & s & x_3 & x_4 & t & \\ \multicolumn{6}{c}{\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)} \end{array}$$

RREF

$x_1 = 2 - s - t$

$x_2 = s$

$x_3 = 2 - t$ OR $\boxed{(2-s-t,\ s,\ 2-t,\ -1,\ t)}$

$x_4 = -1$

$x_5 = t$

If time allows, check your solution by subbing it in to at least one of the original equations.

R_2: $-x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 5$

$-(2-s-t) - s + 2(2-t) - 3(-1) + t = 5$

$-2 \underline{+s} \underline{+t} \underline{-s} + 4 \underline{-2t} + 3 \underline{+t}$

$5 = 5$ ✓

R_3: $x_1 + x_2 - 2x_3 \quad - x_5 = -2$

$(2-s-t) + s - 2(2-t) \quad - t = -2$

$-2 = -2$ ✓

3.(b) $R_1 \leftrightarrow R_3$ $\left(\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 2 & 4 & 2 & 4 \\ 3 & 7 & 2 & 9 \end{array}\right)$ $\begin{array}{l} R_2 \to R_2 - 2\cdot R_1 \\ R_3 \to R_3 - 3\cdot R_1 \end{array}$

$\left(\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & -2 & 4 & -4 \\ 0 & -2 & 5 & -3 \end{array}\right)$ $R_2 \to -\frac{1}{2}\cdot R_2$

$\left(\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & 1 & -2 & 2 \\ 0 & -2 & 5 & -3 \end{array}\right)$ $\begin{array}{l} R_1 \to R_1 - 3\cdot R_2 \\ R_3 \to R_3 + 2\cdot R_2 \end{array}$

$\left(\begin{array}{ccc|c} 1 & 0 & 5 & -2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$ $\begin{array}{l} R_1 \to R_1 - 5\cdot R_3 \\ R_2 \to R_2 + 2\cdot R_3 \end{array}$

$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right)$ $\begin{array}{l} x = -7 \\ y = 4 \\ z = 1 \end{array}$ OR $\boxed{(-7, 4, 1)}$

Geometrical Interpretation

Note: These 3 equations were 3 planes in standard form.
The system is checking where the planes intersect.
We found they intersect at one point $(-7, 4, 1)$.

Inconsistent (No solution)

→ Planes do not intersect at a common place.

Unique Solution

→ One point of Intersection.

Infinite Solutions with 1 Parameter "t"

→ Line of Intersection (You just got parametric equations for that line.)

Infinite Solutions with 2 Parameters "s" and "t"

→ Plane of Intersection (All on the same plane)

Infinite solutions with more than 2 parameters

→ Hyperplane of intersection

If 3 parameters, we would say it is a 3 dimensional hyperplane.

4 parameters → 4 dimensional hyperplane

etc.

3.(c)

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 3 & 0 & 1 \\ -1 & -2 & -2 & 1 & 0 \\ 0 & -1 & -1 & 2 & 1 \end{array}\right) \begin{array}{l} R_2 \to R_2 - 2 \cdot R_1 \\ R_3 \to R_3 + 1 \cdot R_1 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -2 & -1 \\ 0 & -1 & -1 & 2 & 1 \\ 0 & -1 & -1 & 2 & 1 \end{array}\right) \begin{array}{l} R_1 \to R_1 - 1 \cdot R_2 \\ R_3 \to R_3 + 1 \cdot R_2 \\ R_4 \to R_4 + 1 \cdot R_2 \end{array}$$

$$\begin{array}{ccccc} & & s & t & \\ \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 2 \\ 0 & 1 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 = 2 - 3t$

$x_2 = -1 - s + 2t$

$x_3 = s$ OR $(2-3t, -1-s+2t, s, t)$

$x_4 = t$

Check if time allows

3.(d) $R_1 \leftrightarrow R_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 2 & 1 & 1 & 2 \end{array}\right) R_3 \to R_3 - 2 \cdot R_1$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & 0 \end{array}\right) R_3 \to R_3 - 1 \cdot R_2 \quad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Inconsistent, no solution b/c $0 \neq 1$.

Question 3 told us to solve the systems using Gauss-Jordan elimination. Unless specifically told otherwise, this is the method we will always use.

GAUSS-JORDAN ELIMINATION

Step 1: Express the system of equations as an Augmented Matrix.
Step 2: Row-reduce the system to RREF.
Step 3: State the solution to the system

Question 4 tells us to solve the system using Gaussian elimination and back substitution. This method can be quite unpleasant, especially if there are parameters in the solution.

GAUSSIAN ELIMINATION:

Step 1: Express the system as an Augmented Matrix.
Step 2: Row-reduce the system to REF.
Step 3: Use back substitution to complete the solution to the system.

Even if you are told to use Gaussian elimination, use Gauss-Jordan elimination anyway. (A matrix reduced to RREF is also in REF so you've done nothing wrong.) Only if told to use back substitution OR to reduce the matrix to REF, but not RREF, must you use true Gaussian elimination.

4. We have to use true Gaussian elimination here, so row-reduce the system to REF only.

$$\left(\begin{array}{ccc|c} 0 & -1 & 1 & 3 \\ 1 & -1 & -1 & 0 \\ -1 & 0 & -1 & -3 \end{array}\right) R_1 \leftrightarrow R_2 \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & -1 & 1 & 3 \\ -1 & 0 & -1 & -3 \end{array}\right) R_3 \to R_3 + R_1$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & -1 & 1 & 3 \\ 0 & -1 & -2 & -3 \end{array}\right) R_2 \to -R_2 \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & -1 & -3 \\ 0 & -1 & -2 & -3 \end{array}\right)$$ We don't "0" above the leading 1 for REF $R_3 \to R_3 + R_2$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & -3 & -6 \end{array}\right) R_3 \to -\tfrac{1}{3}R_3 \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right)$$ REF ✓

Isolate "x" in Row 1: $x = y + z$
Isolate "y" in Row 2: $y = -3 + z$
Isolate "z" in Row 3: $z = 2$

Only "z" has truly been solved so far. (That's what happens if you only get REF.) But, sub "z = 2" into the "y" equation and you will solve "y". Then, take your values for both "y" and "z" and sub them into the "x" equation to solve "x". That is what we mean by "back substitution".

$x = y + z \to x = -1 + 2 \to x = 1$
$y = -3 + z \to y = -3 + 2 \to y = -1$
$z = 2$

Solution is $x = 1$, $y = -1$, $z = 2$ OR $\boxed{(1, -1, 2)}$

This seems not bad at all, but back substitution can be quite annoying if you have fractions or parameters cluttering up the problem. Some profs like it because they are good at mental math, but I say always use Gauss-Jordan if you can.

A NOTE FOR THOSE OF YOU IN MATH 1310

The textbook used in Math 1310 refers to the "Elimination Method" which is a very stupid and awkward approach unlike the elimination method I teach in Lesson 1. If, God forbid, you are asked to solve a system by elimination in Math 1310 (this has nothing to do with Math 1300), you are being asked to solve a problem using Gaussian elimination but where you never express the system as an Augmented matrix. You leave the variables in the equation.

Again, I stress, this is only if you are in Math 1310, and only if you are told to solve a system by the "Elimination Method" (as opposed to Gauss Jordan elimination or Gaussian elimination). In the unlikely event that those of you in Math 1300 are asked to solve a system by elimination, you would use the good old high school approach I taught in Lesson 1.

For those of you in Math 1310:

If told to solve a system by elimination, first, on scrap paper, row-reduce the system to REF (not RREF). Now, on your exam, rewrite all the steps as equations. Finish the problem with back substitution.

Here is how I would solve Question 4 by "elimination".

First, on scrap paper, I would row-reduce the system to REF just like I already did when using Gaussian elimination. On my test, however, I would write everything in equation form.

For example, the first thing I did earlier was:

$$\left(\begin{array}{ccc|c} 0 & -1 & 1 & 3 \\ 1 & -1 & -1 & 0 \\ -1 & 0 & -1 & -3 \end{array}\right) R_1 \leftrightarrow R_2 \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & -1 & 1 & 3 \\ -1 & 0 & -1 & -3 \end{array}\right)$$

Instead, we will write this:

$$\left\{\begin{array}{rcl} -y+z &=& 3 \\ x-y-z &=& 0 \\ -x \quad\; -z &=& -3 \end{array}\right\} \text{Eqn } 1 \leftrightarrow \text{Eqn } 2 \left\{\begin{array}{rcl} x-y-z &=& 0 \\ -y+z &=& 3 \\ -x \quad\; -z &=& -3 \end{array}\right\}$$

(Note that we call these "equations", "Eqn", rather than rows.) Continue to translate the entire row-reduction we did earlier back into equations:

$$\text{Eqn } 3 \to \text{Eqn } 3 + \text{Eqn } 1 \left\{\begin{array}{rcl} x-y-z &=& 0 \\ -y+z &=& 3 \\ -y-2z &=& -3 \end{array}\right\} \text{Eqn } 2 \to -\text{Eqn } 2$$

$$\begin{Bmatrix} x - y - z = 0 \\ y - z = -3 \\ -y - 2z = -3 \end{Bmatrix} \quad \text{Eqn 3} \rightarrow \text{Eqn 3} + \text{Eqn 2}$$

$$\begin{Bmatrix} x - y - z = 0 \\ y - z = -3 \\ -3z = -6 \end{Bmatrix} \quad \text{Eqn 3} \rightarrow -\tfrac{1}{3} \times \text{Eqn 3}$$

$$\begin{Bmatrix} x - y - z = 0 \\ y - z = -3 \\ z = 2 \end{Bmatrix}$$

Note: I am using French braces "{ }" just for clarity. Your prof might do something different.

Now we can solve by back substitution.

$x = y + z \rightarrow x = -1 + 2 \rightarrow x = 1$

$y = -3 + z \rightarrow y = -3 + 2 \rightarrow y = -1$

$z = 2$

Of course, our solution is the same.

$x = 1, y = -1, z = 2$ OR $\boxed{(1, -1, 2)}$

Personally, I think this elimination method is pretty pointless, but I didn't write your textbook, so don't blame me. This has been on 1310 exams, so be ready to do it. You guys in 1300 can stop smiling. They'll get you later.

5. We can solve both systems simultaneously because they both have the same coefficient matrix. Simply put two columns in the constant matrix to represent the two different systems.

$$\left(\begin{array}{ccc|cc} 1 & 6 & 3 & 34 & 30 \\ 1 & 6 & 2 & 30 & 24 \\ 0 & 2 & 2 & 14 & 16 \end{array}\right) R_2 \to R_2 - R_1 \left(\begin{array}{ccc|cc} 1 & 6 & 3 & 34 & 30 \\ 0 & 0 & -1 & -4 & -6 \\ 0 & 2 & 2 & 14 & 16 \end{array}\right) R_2 \leftrightarrow R_3$$

$$\left(\begin{array}{ccc|cc} 1 & 6 & 3 & 34 & 30 \\ 0 & 2 & 2 & 14 & 16 \\ 0 & 0 & -1 & -4 & -6 \end{array}\right) R_2 \to \tfrac{1}{2} R_2 \left(\begin{array}{ccc|cc} 1 & 6 & 3 & 34 & 30 \\ 0 & 1 & 1 & 7 & 8 \\ 0 & 0 & -1 & -4 & -6 \end{array}\right) R_1 \to R_1 - 6R_2$$

$$\left(\begin{array}{ccc|cc} 1 & 0 & -3 & -8 & -18 \\ 0 & 1 & 1 & 7 & 8 \\ 0 & 0 & -1 & -4 & -6 \end{array}\right) R_3 \to -R_3 \left(\begin{array}{ccc|cc} 1 & 0 & -3 & -8 & -18 \\ 0 & 1 & 1 & 7 & 8 \\ 0 & 0 & 1 & 4 & 6 \end{array}\right) \begin{array}{l} R_1 \to R_1 + 3R_3 \\ R_2 \to R_2 - R_3 \end{array}$$

$$\left(\begin{array}{ccc|cc} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 4 & 6 \end{array}\right)$$

Ignore the fifth column to read off the solution to the first system.

First system: $\left.\begin{array}{l} x = 4 \\ y = 3 \\ z = 4 \end{array}\right\}$ OR $(4,3,4)$

Ignore the fourth column to read off the solutions of the second system.

Second system: $\left.\begin{array}{l} x = 0 \\ y = 2 \\ z = 6 \end{array}\right\}$ OR $(0,2,6)$

$\boxed{\text{The solutions to the two systems are } (4,3,4) \text{ and } (0,2,6) \text{, respectively.}}$

How Many Solutions does a System Have?

You can't really tell how many solutions a system will have until it is in RREF (or at least REF). The Key is to compare the number of leading 1's in your coefficient matrix (ie. the Rank of the coefficient matrix) to the number of variables in your system.

First, is the system consistent?

Recall: (0 0 0 | nonzero) is an inconsistent system (no solution) (ie. If the Rank of the coefficient matrix is less than the Rank of the augmented matrix, the system is inconsistent.)

Then, assuming the system is consistent:

1. A system has a unique solution if and only if you have as many leading 1's as there variables (if the Rank = the number of variables).
2. A system has infinite solutions if there are more variables than leading 1's. (if the Rank < number of variables).

eg. If a system has 3 variables, it must have a rank of 3 (3 leading 1's) in its coefficient matrix in order to have a unique solution (assuming the system is consistent, of course). If its rank is only 2, it will have one parameter (infinite solutions). If it rank is only 1, it will have two parameters (infinite solutions).

If you look back at the systems we solved in question 3, notice:

The augmented matrix for 3.(a) is

$$\left(\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 2 \\ -1 & -1 & 2 & -3 & 1 & 5 \\ 1 & 1 & -2 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

I can immediately see this system cannot have a unique solution. It only has 4 Rows so, even assuming each row ended up with a leading 1 after row-reduction, the best it can do is 4 leading 1's (Rank of 4). <u>But</u>, it has 5 variables. Thus, it is guaranteed to end up with at least one parameter. Of course, the system could even be inconsistent and have no solution at all.

In fact, the RREF of this system is

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$ as we found above in 3.(a)

5 variables but the Rank = 3, so there will be 5 - 3 = 2 parameters (infinite solutions).

3.(b) had the augmented matrix

$$\left(\begin{array}{ccc|c} 3 & 7 & 2 & 9 \\ 2 & 4 & 2 & 4 \\ 1 & 3 & -1 & 4 \end{array}\right)$$. This matrix has 3 Rows, so it could conceivably end up with a Rank of 3. There are 3 variables so, if its Rank is 3, it will have a unique solution. If its Rank is less than 3, the system could end up being inconsistent (no solution) or it will have parameters (infinite solutions). We don't know what will be the case until it is row-reduced. In fact, we found the system reduced to

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right)$$ which is Rank 3. It does have a unique solution.

Note: You never know for sure how many solutions a system will have until you have row-reduced it and identified its Rank.

3(d) had a similar format to 3(b)
Its augmented matrix is $\left(\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & 1 \end{array}\right)$,
so it, too, could have a Rank of 3 (because it has 3 Rows).

3 Variables → If Rank = 3 → unique solution
→ If Rank < 3 → either infinite solutions or none.

In fact, it row-reduced to

$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right)$ an inconsistent system

(Rank of coefficient matrix = 2
Rank of augmented matrix = 3).

Some systems can immediately rule out a unique solution (their rank could never equal the number of variables in the system because there aren't enough rows). But, that's about all we can say before a system is row-reduced.

Homogeneous Systems

If every single equation in a system equals 0, the system is homogeneous.

eg.
$$\left.\begin{aligned} x + y + z &= 0 \\ 2x - y + 3z &= 0 \\ x + 2y + 5z &= 0 \end{aligned}\right\}$$
is a homogeneous system (all 3 equations = 0)

The augmented matrix would be:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & 3 & 0 \\ 1 & 2 & 5 & 0 \end{array}\right)$$

Tip: A "0" column (like column 4 above) will never change as you row-reduce.

That means a homogeneous system will never reduce to $(0\ 0\ 0 \mid \text{nonzero})$ because the last column will always be 0.

Homogeneous systems will always be consistent. We still must establish the Rank of the matrix before we know if the system will have a unique solution or infinite solutions.

Let's get back to the questions.

6. If you are ever asked how many solutions a system has, make sure you row-reduce it first. (You can go for RREF or REF.) If they ever introduce letters like "K" and ask you to find "K" to achieve a certain number of solutions:

First, Row-reduce the augmented matrix. (But, don't bother to row-reduce the last row. Get all your leading 1's, etc as usual, but, when it comes time to get the leading 1 in the last row, QUIT!)

Then, Focus on that last row and subject it to the three cases I outline below. Case 1 guarantees an inconsistent system. Cases 2 and 3 guarantee a consistent system, but you must compare the Rank to the number of variables to see how many solutions the system would have.

<u>Case 1</u>: Can you choose value(s) of K, or whatever, to make that last row $\boxed{(0\ 0\ 0 \mid \text{nonzero})}$?

If so, as always, those values of K will cause the system to have <u>no</u> solution (the system is inconsistent).

<u>Case 2</u>: Are there value(s) of K, or whatever, that make the last row <u>all</u> 0's → $\boxed{(0\ 0\ 0 \mid 0)}$?

Then, the system is <u>consistent</u> but compare the Rank of the coefficient matrix to the number of variables to decide if it is a <u>unique</u> solution or <u>infinite</u> solution.

<u>Case 3</u>: Are there value(s) of K that make the last row

$\boxed{(0\ 0\ \text{nonzero} \mid \text{irrelevant})}$

ie. Can you get a nonzero in the coefficient matrix. That means you have added 1 more to the Rank of the matrix. Again, as in Case 2, compare the Rank to the number of variables to determine the number of solutions (unique or infinite).

6. The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & K \\ -1 & 2 & -3 & 1 \end{array}\right) R_3 \to R_3 + R_1$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & K \\ 0 & 1 & -1 & 1 \end{array}\right) \begin{array}{l} R_1 \to R_1 + R_2 \\ \\ R_3 \to R_3 - R_2 \end{array}$$ (I could just get REF and leave R_1 alone.)

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & K \\ 0 & 1 & -1 & K \\ 0 & 0 & 0 & -K+1 \end{array}\right)$$ QUIT (Don't get a leading 1 in the last row.)

Subject that last row $(0\ 0\ 0 | -K+1)$ to the three cases. (It is of no importance that there are K's in rows 1 and 2. As long as you have row-reduced those rows, the last row is all that matters in determining the number of solutions.)

<u>Case I</u>: Can we make

$$\begin{pmatrix} 0 & 0 & 0 & | & -K+1 \end{pmatrix} \text{ into } \begin{pmatrix} 0 & 0 & 0 & | & \text{nonzero} \end{pmatrix}?$$
$$\begin{matrix} 0 & 0 & 0 & \text{nonzero} \end{matrix}$$

Obviously! Make "$-K+1$" nonzero ($\neq 0$)

ie $-K+1 \neq 0 \to K \neq 1$ satisfies <u>Case I</u>.

$$\boxed{\text{If } K \neq 1\text{, the system has } \underline{\text{no}} \text{ solution.}}$$

Case 2: Can we make $(0\ 0\ 0 | -K+1)$ into $(0\ 0\ 0 | 0)$?

Obviously! Make "$-K+1=0$" $\to K=1$

If $K=1$, the system has Rank $=2$ (2 leading 1's in Rows 1 and 2), but it has 3 variables $\to$ infinite solutions (with one parameter to be precise.)

$\boxed{\text{If } K=1\text{, there are infinite solutions.}}$

Case 3: Can we make $(0\ 0\ 0 | -K+1)$ into $(0\ 0\ \text{nonzero} | \text{irrelevant})$?

NO! We have a "0" row in the coefficient matrix so we'll never be able to change that. No value of K will satisfy Case 3.

Answering their questions then:

6. (a) If $K=1$, the system has infinite solutions.

(b) No value of K will cause a unique solution.

(c) If $K \neq 1$, the system has no solution.

7. Again, row-reduce but don't get a leading 1 in the last row.

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & a & b \end{array}\right) R_3 \to R_3 - 2R_2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & a-2 & b-4 \end{array}\right) \text{ Quit!}$$

Subject $(0 \ 0 \ a-2 \,|\, b-4)$ to the three cases.

Case 1: $(0 \ 0 \ \underset{a-2}{0} \,|\, \underset{b-4}{\text{nonzero}})$

Let $a-2=0$ and $b-4 \neq 0$.

$\boxed{\text{If } a=2 \text{ and } b \neq 4\text{, the system is inconsistent and has no solution.}}$

Case 2: $(0 \ 0 \ \underset{a-2}{0} \,|\, \underset{b-4}{0})$

Let $a-2=0$ and $b-4=0$.

Then, the system is consistent with Rank = 2 (2 leading 1's, one in R_1 and one in R_2). But, there are 3 variables, so the system will have one parameter (3 variables − Rank of 2 = 1 parameter)

$\boxed{\text{If } a=2 \text{ and } b=4\text{, the system has infinite solutions with one parameter.}}$

Case 3: $\left(0 \quad 0 \quad \underset{a-2}{\text{nonzero}} \;\middle|\; \underset{b-4}{\text{irrelevant}}\right)$

If $a-2 \neq 0$, then $b-4$ could be anything. The system would then have a Rank of 3 (the nonzero in R_3 is guaranteed to become a 3rd leading 1 if we continued to row-reduce).
3 variables, Rank 3 $\rightarrow$ unique solution

If $a \neq 2$, the system would have a unique solution (b could be any real number).

Answering their questions:

7. (a) The system has no solution if $a = 2$ and $b \neq 4$.

(b) The system has a unique solution if $a \neq 2$.

(c) The system has infinitely many solutions if $a = 2$ and $b = 4$.

Do not make the mistake of thinking Case 2 always makes infinite solution and Case 3 always makes a unique solution. All we can say is both Case 2 and Case 3 make consistent systems. The Rank tells you how many solutions.

8. $\left(\begin{array}{ccc|c} 1 & 0 & a+1 & 7 \\ 0 & 1 & -5 & 6 \\ 0 & 0 & a^2-4a & a-4 \end{array}\right)$ Already Row-reduced except for the last Row.

Case 1: $(0 \quad 0 \quad 0 \mid \text{nonzero})$

$\qquad\qquad\qquad\quad a^2-4a \quad a-4$

$\underbrace{a^2-4a}_{\text{Factor out "a"}} = 0$ and $a-4 \neq 0$

$a(a-4) = 0$

$\underline{a=0}$ OR $a-4=0$

$\underline{a=4}$ but $\underline{a \neq 4}$

Obviously a can't $=4$ and $\neq 4$ at the same time. Put this all together and we come up with:

If $a=0$ and $a \neq 4$ we will get $(0 \quad 0 \quad 0 \mid \text{nonzero})$. But, that really boils down to $a=0$ will be the only way to get $(0\ 0\ 0 \mid \text{nonzero})$ (specifically, it creates $(0\ 0\ 0 \mid 4)$.

If a is any other number, you would end up with $(0\ 0 \text{ nonzero} \mid \text{etc.})$ (Case 3). Exception: if $a=4$, you end up with $(0\ 0\ 0 \mid 0) \rightarrow$ Case 2.

Summarizing:

If $a=0$, we get $(0\ 0\ 0|4)$ causing no solution (inconsistent).

If $a=4$, we get $(0\ 0\ 0|0)$, a consistent system of Rank = 2. With 3 Variables and Rank 2, the system has infinite solutions with one parameter.

If $a \neq 0$ and $a \neq 4$ (ie. if a is any other number besides 0 or 4) we get $(0\ 0\ \text{nonzero}|\text{irrelevant})$, a consistent system with Rank=3. 3 Variables, Rank 3 $\rightarrow$ unique solution.

<u>Answers</u>

8. (a) The system has infinite solutions if $a=4$.

(b) The system has no solution if $a=0$.

(c) The system has a unique solution if $a \neq 0$ <u>and</u> $a \neq 4$.

WORD PROBLEMS

It is highly unlikely you will have a word problem in a Math 1300 exam (it hasn't happened in decades to my knowledge). It might happen on a exam for the Distance version, though (and is on the hand-in assignments). Those of you in Math 1310 must certainly be able to do word problems and be ready for them on exams.

There is nothing to fear! Thanks to matrices it is quite easy to set up a Word Problem.

Each problem will break up the info into 2 categories. Each of those categories is then split into two or more "levels". The levels of one category will be the columns of your matrix, and the levels of the other category will be the rows.

Key: The columns must always represent the unknowns you want to solve. Make sure you figure out which are the columns and which are the rows.

> Tip: You will always be given Totals for the levels of one of the categories. The levels that have Totals will become the ROWS of your matrix. Obviously, the other category's levels will be the columns of your matrix (and will become the unknowns or variables in your system).

Your matrix will look something like this (the number of rows and columns will vary, of course).

eg. Category 1 has 2 levels (A and B) and we were given the Total amount of A and Total amount of B.

Category 2 has 3 levels (P, Q and R)

Therefore:

$$\begin{array}{lll} & & \quad\text{Category 2} \\ & & \begin{array}{ccc|c} P & Q & R & \text{Totals} \end{array} \\ \text{Category 1} & \begin{cases} A \\ B \end{cases} & \left(\begin{array}{ccc|c} & & & \text{given} \\ & & & \text{given} \end{array}\right) \end{array}$$

They will then give you all the numbers to feed into the matrix. Just match the number to the proper location.

<u>Important</u>: Once you define the variables you are going to use (let's say x_1, x_2 and x_3) then always state the variables are ≥ 0 ($x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$).

Unlike math problems where are solutions can be any kind of numbers (positive, negative, 0, whatever), word problems are generally dealing with practical things that could never be negative.

x_1 might be the number of chickens. You can't have a negative amount of chickens! I bought −3 chickens (huh?). That's why $x_1 \geq 0$. You could be 0 chickens, 1 chicken, 2, etc. You could even buy $\frac{1}{2}$ a chicken (if you're eating it; not as a pet).

Then, keep this limitation in mind when you state your solution.

9. We have 3 People (Anne, Betty, Carol) and 3 Kinds of Fruit (Apples, Bananas, and Grapes). We were told the total price Anne, Betty and Carol paid so they will be the rows.

	Apples	Bananas	Grapes	Total
Anne	1	2	0	1.85
Betty	2	0	1	3.65
Carol	0	1	2	3.95

Note: I put $(1 \quad 2 \quad 0 \mid 1.85)$ in Anne's Row because we were told Anne bought 1 pound of apples (I put 1 in the Anne, Apple cell), 2 pounds of bananas (I put 2 in the Anne, Banana cell), she bought no grapes (0 in the Anne, Grape cell) and paid a Total of \$1.85.

Define your variables (and make sure you tell the marker the variables are ≥ 0 or you will lose marks!).
Don't say a = apples! Be more precise
Here, a = price per pound for apples.
(Remember, the columns are the variables.)

Of course, you could use any letters you want. I'll use:

a = price per pound for apples
b = price per pound for bananas
g = price per pound for grapes.

$a \geq 0$, $b \geq 0$, $g \geq 0$
(you can't have negative prices.)

If they ask you to set up a system of equations for the problem (they didn't here), simply translate the augmented matrix above into the actual equations:

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1.85 \\ 2 & 0 & 1 & 3.65 \\ 0 & 1 & 2 & 3.95 \end{array}\right) \to \begin{cases} a + 2b = 1.85 \\ 2a + g = 3.65 \\ b + 2g = 3.95 \end{cases}$$

I will solve the system by row-reduction (The Gauss-Jordan elimination method if you want to talk like a prof.)

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1.85 \\ 2 & 0 & 1 & 3.65 \\ 0 & 1 & 2 & 3.95 \end{array}\right) R_2 \to R_2 - 2R_1$$

(Note: $3.65 - 2 \times 1.85$
$= 3.65$
$- 3.70$
$\boxed{-0.05}$)

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1.85 \\ 0 & -4 & 1 & -.05 \\ 0 & 1 & 2 & 3.95 \end{array}\right) R_2 \Leftrightarrow R_3$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1.85 \\ 0 & 1 & 2 & 3.95 \\ 0 & -4 & 1 & -.05 \end{array}\right) \begin{array}{l} R_1 \to R_1 - 2R_2 \\ R_3 \to R_3 + 4R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -4 & -6.05 \\ 0 & 1 & 2 & 3.95 \\ 0 & 0 & 9 & 15.75 \end{array}\right) R_3 \to \frac{1}{9} R_3$$

$\left(\underline{\text{Note:}}\ \frac{15.75}{9} = 1.75\right)$

$$\left(\begin{array}{ccc|c} 1 & 0 & -4 & -6.05 \\ 0 & 1 & 2 & 3.95 \\ 0 & 0 & 1 & 1.75 \end{array}\right) \begin{array}{l} R_1 \to R_1 + 4R_3 \\ R_2 \to R_2 - 2R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0.95 \\ 0 & 1 & 0 & 0.45 \\ 0 & 0 & 1 & 1.75 \end{array}\right)$$

Therefore: $a = 0.95$
$b = 0.45$
$g = 1.75$

Always give your solution to a word problem in the form of a sentence.

Apples are \$0.95 per pound, bananas \$0.45 per pound, and grapes \$1.75 per pound.

Sometimes there is no avoiding fractions when you are row-reducing. Your prof may show you some fancy maneouver using a different row operation or two to avoid a fraction, but try this at your peril. There is too much time wasted in thinking (and it might not work anyway). DON'T TRY IT! (If you must, look at 9.(n) in the Practise Problems at the end of this lesson to see an example of avoiding fractions.) I say you will probably be faster if you just deal with the fractions.

How to ADD/SUBTRACT Fraction:

① $\dfrac{A}{B} \pm \dfrac{C}{D}$ ② $= \dfrac{A \cdot D \pm B \cdot C}{B \cdot D}$ } Multiply the diagional to get the Top

Multiply the Bottoms together

eg. $\dfrac{2}{3} + \dfrac{4}{5} = \dfrac{10 + 12}{15} = \dfrac{22}{15}$

$4 \cdot 3 = 12$ $\quad 3 \cdot 5 = 15$ $\quad 2 \cdot 5 = 10$

eg. $\dfrac{2}{3} - \dfrac{1}{7} = \dfrac{14 - 3}{21} = \dfrac{11}{21}$

eg. $\dfrac{2}{3} - \dfrac{13}{10} = \dfrac{20 - 39}{30} = \dfrac{-19}{30}$

eg. $\dfrac{2}{5} + \dfrac{3}{10} = \dfrac{20 + 15}{50} = \dfrac{35}{50} = \dfrac{7}{10}$

10. Trucks (1, 2 & 3) vs Machine (A & B)
Total of 18 of A, 26 of B → ROWS!

$$\begin{array}{r|ccc|c} & \#1 \ (x_1) & \#2 \ (x_2) & \#3 \ (x_3) & \text{Totals} \\ \hline \text{Machine A} & 2 & 1 & 3 & 18 \\ \text{Machine B} & 3 & 2 & 4 & 26 \end{array}$$

(Trucks)

x_1 = Number of Truck #1's used
x_2 = " " Truck #2's "
x_3 = " " Truck #3's "

$x_1 \leq 4$, $x_2 \leq 4$, $x_3 \leq 4$
$x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 18 \\ 3 & 2 & 4 & 26 \end{array}\right) R_1 \to \tfrac{1}{2} R_1 \left(\begin{array}{ccc|c} 1 & 1/2 & 3/2 & 9 \\ 3 & 2 & 4 & 26 \end{array}\right)$$

$R_2 \to R_2 - 3 \cdot R_1$

$-\frac{3}{2} + 2 = \frac{-3+4}{2} = \frac{1}{2}$

$-\frac{9}{2} + 4 = \frac{-9+8}{2} = \frac{-1}{2}$

$$\left(\begin{array}{ccc|c} 1 & 1/2 & 3/2 & 9 \\ 0 & 1/2 & -1/2 & -1 \end{array}\right) R_2 \to 2 \cdot R_2$$

reciprocal

$$\left(\begin{array}{ccc|c} 1 & 1/2 & 3/2 & 9 \\ 0 & 1 & -1 & -2 \end{array}\right) R_1 \to R_1 - \tfrac{1}{2} \cdot R_2 \begin{array}{c} \\ \left(\begin{array}{ccc|c} 1 & 0 & 2 & 10 \\ 0 & 1 & -1 & -2 \end{array}\right) \end{array}$$

t

$\frac{1}{2} + \frac{3}{2} = \frac{4}{2} = 2$

$$\boxed{\begin{aligned} x_1 &= 10 - 2t \\ x_2 &= -2 + t \\ x_3 &= t \end{aligned}}$$

WORD PROBLEM → give all feasible solutions
$x_i \geq 0$, $x_i \leq 4$

Sub in $t=0$, $t=1$, etc to generate actual solution! **Note** since $x_3 = t$ and since $x_3 \geq 0$, $t \geq 0$

<u>$t=0$</u>

$x_1 = 10-2(0) = 10$ <u>impossible</u> $x_1 \leq 4$

<u>$t=1$</u>

$x_1 = 10-2(1) = 8$ <u>impossible</u>

<u>$t=2$</u>

$x_1 = 10-2(2) = 6$ No good

<u>$t=3$</u>

$$\left.\begin{aligned} x_1 &= 10-2(3) = \underline{\underline{4}} \\ x_2 &= -2+3 = \underline{\underline{1}} \\ x_3 &= \underline{\underline{3}} \end{aligned}\right\} \boxed{(4,1,3)}$$

<u>$t=4$</u>

$$\left.\begin{aligned} x_1 &= 10-2(4) = \underline{\underline{2}} \\ x_2 &= -2+4 = \underline{\underline{2}} \\ x_3 &= \underline{\underline{4}} \end{aligned}\right\} \boxed{(2,2,4)}$$

$t=5$ & on causes $x_3 = 5$ & on No good!

$\therefore$ We can use 4 of Truck #1, 1 of Truck #2 and 3 of Truck #3

<u>OR</u> 2 of Truck #1, 2 of Truck #2 and 4 of Truck #3.

11. This is a logic problem.

First A Zero Matrix (a matrix consisting of strictly 0's) is in RREF. (There is no requirement that you have to have nonzero numbers. Only if there are nonzero numbers must you have leading 1's, etc. to be in RREF.)

3×2 means 3 rows by 2 columns

So $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a 3×2 matrix in RREF.

Now, start introducing leading 1's.

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ is in RREF. In fact, $\begin{pmatrix} 1 & K \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ where K is any real number works.

Note: something like $\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$ is no good (zero rows must be below leading 1 rows. You can't have a leading 1 in R_2 if you have a zero row in R_1).

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ is in RREF. There is no law that says the Top left corner must be the first leading 1 (it is just unusual for that to not be the case).

That exhausts all the possibilities where there is a leading 1 only in R_1.
We could also have leading 1's in both R_1 and R_2. (Remember, though, leading 1's must be deeper in the rows as you go down.)

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ is in RREF but that is the only way that would work if we have two leading 1's.
It is impossible to have 3 leading 1's because they wouldn't be able to get deeper in each time.
Again, we can't have leading 1's in R_1 and R_3, for example because that would mean a 0-row is above a leading 1 row (a no-no for RREF).

<u>Summarizing</u> All 3×2 RREF matrices are:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & K \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

where K = any real number.

12. This question looks scary because there are no numbers, just a gaggle of letters. RELAX! You are told (x_1, y_1, z_1) is a solution to $ax + by + cz = d$ so that means you can sub in $x = x_1$, $y = y_1$, $z = z_1$

We Know: $\boxed{ax_1 + by_1 + cz_1 = d}$

Similarly, we are told (x_2, y_2, z_2) is a solution to equation (1).

We Know: $\boxed{ax_2 + by_2 + cz_2 = d}$

Finally, we are told (x_0, y_0, z_0) is a solution to equation (2).

We Know: $\boxed{ax_0 + by_0 + cz_0 = 0}$

Keep these three facts in mind as you attack the questions.

12. (a) We have to show $(x_1 - x_2, y_1 - y_2, z_1 - z_2)$ is a solution to equation (2).

ie, If we sub $x = x_1 - x_2$, $y = y_1 - y_2$, $z = z_1 - z_2$ into $ax + by + cz = 0$ will it prove true?

Focus on the Left Hand Side (LHS) and see if it simplifies into "0", the Right Hand Side.

$LHS = ax + by + cz \rightarrow$ sub in $(x_1 - x_2, y_1 - y_2, z_1 - z_2)$

$= a(x_1 - x_2) + b(y_1 - y_2) + c(z_1 - z_2)$

Multiply everything to remove the brackets.

$LHS = ax_1 - ax_2 + by_1 - by_2 + cz_1 - cz_2$

Separate the x_1, y_1, z_1 terms from the x_2, y_2, z_2 terms.

$LHS = ax_1 + by_1 + cz_1 - ax_2 - by_2 - cz_2$

Factor the "−" sign out of those last three terms.

$$LHS = \underbrace{ax_1 + by_1 + cz_1}_{\substack{\text{We know} \\ ax_1 + by_1 + cz_1 = d}} - (\underbrace{ax_2 + by_2 + cz_2}_{\substack{\text{We know} \\ ax_2 + by_2 + cz_2 = d}})$$

$LHS = d - d \rightarrow \boxed{LHS = 0}$

We have proven
$a(x_1 - x_2) + b(y_1 - y_2) + c(z_1 - z_2) = 0$
Proving $(x_1 - x_2, y_1 - y_2, z_1 - z_2)$ is a solution to equation 2.

12(b) Attack this problem the same way.

We must prove $(x_1 - x_0, y_1 - y_0, z_1 - z_0)$ is a solution to $ax + by + cz = d$

$LHS = ax + by + cz$

Sub in $x = x_1 - x_0$, $y = y_1 - y_0$, $z = z_1 - z_0$

$LHS = a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)$

$LHS = ax_1 - ax_0 + by_1 - by_0 + cz_1 - cz_0$

Separate the x_1, y_1, z_1 terms from the x_0, y_0, z_0 terms.

$LHS = ax_1 + by_1 + cz_1 - ax_0 - by_0 - cz_0$

$$LHS = \underbrace{ax_1 + by_1 + cz_1}_{\substack{\text{We know} \\ ax_1 + by_1 + cz_1 = d}} - (\underbrace{ax_0 + by_0 + cz_0}_{\substack{\text{We know} \\ ax_0 + by_0 + cz_0 = 0}})$$

$LHS = d - 0 \rightarrow \boxed{LHS = d}$

We have proven

$$a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0) = d$$

proving $(x_1 - x_0, y_1 - y_0, z_1 - z_0)$ is a solution to equation (1).

12.(c) Prove (Kx_0, Ky_0, Kz_0) is a solution to $ax + by + cz = 0$

$LHS = ax + by + cz$

Sub in $x = Kx_0$, $y = Ky_0$, $z = Kz_0$

$LHS = a(Kx_0) + b(Ky_0) + c(Kz_0)$

Tidy this up.

$LHS = aKx_0 + bKy_0 + cKz_0$

"K" is a common factor.
Factor "K" out.

$LHS = K(\underbrace{ax_0 + by_0 + cz_0}_{})$

We know $ax_0 + by_0 + cz_0 = 0$

$LHS = K(0) \longrightarrow \boxed{LHS = 0}$

We have proven

$$a(Kx_0) + b(Ky_0) + c(Kz_0) = 0$$

proving (Kx_0, Ky_0, Kz_0) is a solution to $ax + by + cz = 0$.

Homework:

- Memorize the facts about **The Rank of a Matrix** on page 37.
- Study the lesson thoroughly until you can do all of **questions 1 to 12** on pages 37 to 40 from start to finish without any assistance.
- Do all of the **Practise Problems** below (solutions are on pages 97 to 101).

Practise Problems:

1. In each of the following parts (a) – (c) of this question, you are given the row-reduced echelon form of the augmented matrix of a system of linear equations. In each case, say whether the system is inconsistent, has a unique solution, or has infinitely many solutions. If the system is inconsistent, explain why. If the system is consistent, state the solution(s).

(a) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right)$

(b) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$

(c) $\left(\begin{array}{cccc|c} 1 & 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

2. **(a)** Write the augmented matrix for the system:

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & + & x_3 & + & 2x_4 & = & -3 \\ -2x_1 & + & 4x_2 & & & - & x_4 & = & 1 \\ & & & & x_3 & + & 2x_4 & = & -4 \end{array}$$

(b) Find the reduced row-echelon form for the augmented matrix in (a).

(c) Write all solutions to the system in (a).

3. A system of linear equations is given by $\begin{array}{rcrcrcr} x & - & y & + & 3z & = & -1 \\ -x & + & y & - & 2z & = & 0 \\ 3x & - & 3y & + & 5z & = & 1 \\ x & - & y & & & = & 2 \end{array}$.

(a) Find the reduced row echelon form of the augmented matrix of the system.

(b) Write all the solutions to the system.

(c) Find the solution set which has $x = 3$.

4. The augmented matrix of a system of linear equations has the following row-reduced echelon form matrix:

$$R=\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(a) How many equations are there in the original system of linear equations?

(b) How many variables does the system of linear equations contain?

(c) How many solutions does the system have? Explain.

(d) How many leading 1's does R contain?

(e) What is the rank of the coefficient matrix?

5. Determine the number of solutions the following linear system of homogeneous equations has **without** solving the system. Give a reason for your answer.

$$\begin{array}{rcrcrcrcl} 3x_1 & + & 5x_2 & - & 7x_3 & + & x_4 & = & 0 \\ x_1 & - & 3x_2 & + & x_3 & & & = & 0 \end{array}$$

6. Each matrix A and B below is the augmented matrix of a system of linear equations in x_1, x_2, x_3, and x_4. For **each** of A and B do the following.

(i) Put the matrix into reduced row echelon form, while stating exactly which elementary row operations you are using.

(ii) State **how many** solutions the system has: none, one, or infinitely many.

(iii) If the system has solutions, give the **general solution** in vector form, using parameters *s, t, u, v,...* (as necessary).

(a) $A=\left[\begin{array}{cccc|c} 1 & 2 & 1 & -3 & 2 \\ 0 & 0 & 1 & 1 & 4 \\ 2 & 4 & 2 & -6 & 4 \end{array}\right]$

(b) $B=\left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & -5 \\ 0 & 1 & 0 & 1 & -4 \\ 2 & 4 & 4 & 6 & 5 \end{array}\right]$

7. Given the system
$$\begin{array}{rcrcrcrcr} 2x_1 & - & 4x_2 & + & 2x_3 & - & 2x_4 & = & 2 \\ 2x_1 & - & 4x_2 & + & 3x_3 & - & 3x_4 & = & 2 \\ 4x_1 & - & 8x_2 & + & 3x_3 & - & 4x_4 & = & 5 \\ & & & - & x_3 & + & x_4 & = & 0 \end{array}.$$

(a) Solve the system of equations above by completely reducing the augmented matrix to row-reduced echelon form.

(b) Interpret the solution geometrically (e.g., as a point, line, plane, hyperplane, etc.).

(c) What is the rank of the augmented matrix?

8. Solve the following systems of equations using Gauss-Jordan or Gaussian elimination:

(a)
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & 4 \\ 3x_1 & + & 4x_2 & - & 2x_3 & = & 7 \end{array}$$

(b)
$$\begin{array}{rcrcrcr} x_1 & + & x_2 & - & x_3 & = & 7 \\ 4x_1 & - & x_2 & + & 5x_3 & = & 4 \\ 2x_1 & + & 2x_2 & - & 3x_3 & = & 0 \end{array}$$

(c)
$$\begin{array}{rcrcrcr} x & + & y & - & z & = & -1 \\ x & - & y & + & 5z & = & 5 \\ 2x & + & 3y & - & 5z & = & -5 \end{array}$$

(d)
$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 1 \\ 2x & + & y & + & 3z & = & 1 \\ x & + & y & + & 2z & = & 1 \end{array}$$

(e)
$$\begin{array}{rcrcrcrcrcr} x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & 7 \\ 2x_1 & + & 2x_2 & + & 3x_3 & + & 3x_4 & + & 4x_5 & = & 18 \end{array}$$

(f)
$$\begin{array}{rcrcrcrcr} 2x_1 & + & 2x_2 & - & x_3 & & & = & 0 \\ -x_1 & - & x_2 & + & 2x_3 & - & 3x_4 & = & 0 \\ x_1 & + & x_2 & - & 2x_3 & & & = & 0 \\ & & & & x_3 & + & x_4 & = & 0 \end{array}$$

(g)
$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & & & + & 2x_4 & = & 5 \\ 2x_1 & + & 4x_2 & + & 2x_3 & + & 4x_4 & = & 12 \\ -x_1 & - & 2x_2 & + & 3x_3 & - & 2x_4 & = & -2 \\ & & x_2 & & & - & 3x_4 & = & 0 \end{array}$$

(h)
$$\begin{array}{rcrcrcr} x & + & y & - & z & = & 1 \\ 2x & + & y & + & 2z & = & 5 \\ 4x & + & 3y & & & = & 7 \end{array}$$

9. Solve the following systems of equations using Gauss-Jordan or Gaussian elimination:

(a)
$$\begin{aligned} x_1 + 2x_2 + 2x_4 &= 4 \\ 2x_1 + 4x_2 + 2x_3 + 5x_4 &= 2 \\ 3x_1 + 6x_2 + 2x_3 + 7x_4 &= 6 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 2 \\ x_1 + x_2 + x_3 + 2x_4 + 2x_5 &= 3 \\ x_1 + x_2 + x_3 + 2x_4 + 3x_5 &= 2 \end{aligned}$$

(c)
$$\begin{aligned} x + y &= 6 - z \\ z - y &= 2 - x \\ x - z &= 4 - y \end{aligned}$$

(d)
$$\begin{aligned} x - y &= 5 \\ 3x + y - z &= 3 \\ 2y - x + z &= -8 \end{aligned}$$

(e)
$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 + 2x_4 &= 2 \\ x_1 + x_3 &= 3 \\ 2x_1 + x_2 + 2x_3 + x_4 &= 4 \end{aligned}$$

(f)
$$\begin{aligned} 3x_2 + 3x_3 &= -3 \\ x_1 - 2x_2 - 3x_3 &= 2 \\ -x_1 + 4x_2 + 6x_3 &= 8 \end{aligned}$$

(g)
$$\begin{aligned} 3x_1 + 3x_2 + 6x_3 &= 3 \\ 2x_1 + x_2 - x_3 &= 3 \\ - x_2 - 5x_3 &= 1 \\ -5x_1 - 2x_2 + 5x_3 &= -8 \end{aligned}$$

9. (Continued) Solve the following systems of equations using Gauss-Jordan or Gaussian elimination:

(h)
$$\begin{aligned} x_1 + 2x_2 - 5x_3 + x_4 &= 0 \\ 2x_1 - 2x_2 + 8x_3 + 8x_4 &= 6 \\ 3x_1 + 2x_2 - 3x_3 + 7x_4 &= 4 \end{aligned}$$

(i)
$$\begin{aligned} x_1 + x_2 - x_3 + x_4 &= 2 \\ -2x_1 + x_3 &= -4 \\ x_1 - x_2 - 2x_3 + 2x_4 &= -1 \end{aligned}$$

(j)
$$\begin{aligned} x_1 + 3x_2 - x_3 + x_4 - 2x_5 &= 8 \\ x_1 + 6x_2 - 4x_3 + 2x_4 - x_5 &= 14 \\ 2x_1 + 8x_2 - 4x_3 + 3x_4 - 4x_5 &= 21 \end{aligned}$$

(k)
$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 2 \\ x_1 + x_2 + x_3 + 2x_4 + 2x_5 &= 3 \\ x_1 + x_2 + x_3 + 2x_4 + 3x_5 &= 2 \end{aligned}$$

(l)
$$\begin{aligned} x_1 + 2x_4 &= 3 \\ -2x_1 + x_2 + x_3 - 4x_4 &= -7 \\ x_3 - x_4 &= 1 \\ -2x_1 + x_2 - x_3 - 2x_4 &= -9 \end{aligned}$$

(m)
$$\begin{aligned} 7x_1 + 3x_2 + x_3 &= 0 \\ 7x_1 + 3x_2 + x_4 + x_5 &= 1 \end{aligned}$$

(n)
$$\begin{aligned} -40x + 16y + 9z &= 1 \\ 13x - 5y - 3z &= -2 \\ 5x - 2y - z &= -1 \end{aligned}$$

10. Given the following augmented matrix for a system of linear equations:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & k+1 & 0 & 0 \\ 0 & 0 & k & 5 \end{array}\right).$$

For what value(s) of k, if any, are there:

(a) a unique solution?

(b) infinitely many solutions?

(c) no solution?

Give reasons for your answers.

11. Find all c such that the system below has no solutions.

$$\begin{aligned} x - 2y &= 1 \\ \frac{1}{2}x - y &= c \end{aligned}$$

12. Find a value of p and q so that the system

$$\begin{aligned} x \quad\quad + z &= 1 \\ y + z &= 1 \\ py + qz &= 1 \end{aligned}$$

(a) has a unique solution.

(b) has an infinite number of solutions.

(c) has no solution.

13. Let $R = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & a-2 & b \\ 0 & 0 & 0 & 0 \end{array}\right)$ be a *row-echelon* form of the augmented matrix of a linear system.

(a) What are the number of equations and the number of variables in the system?

(b) Find all of the values of a and b for which the system has a unique solution.

(c) Find all of the values of a and b for which the system has no solution.

(d) Find all of the values of a and b for which the system has infinitely many solutions. How many parameters are there in the solution set?

14. Let $\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b+1 \end{array}\right)$ be a row-echelon form of the augmented matrix of a linear system.

(a) What are the number of equations and the number of variables in the system?

(b) Find all values of a and b such that the system has no solutions.

(c) Find all of the values of a and b such that the system has a unique solution.

15. Let $A = \left(\begin{array}{ccc|c} 2 & -3 & 5 & a \\ 1 & -1 & 2 & 0 \\ -4 & 6 & -10 & 1 \end{array}\right)$ be the augmented matrix of a linear system.

(a) What are the number of equations and the number of variables in the system?

(b) Find all values of "a" for which the system has no solution.

(c) Find all of the values of "a" for which the system has a infinitely many solutions.

16. Let the augmented matrix of a linear system be given by $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & x & y \end{array}\right)$.

For what values of x and y is there

(a) No solution?

(b) Exactly one solution?

(c) Infinitely many solutions?

17. Suppose that the augmented matrix of a linear system of equations is given by

$$\left[\begin{array}{rrr|r} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -5 \\ 2 & -5 & k & 3k+5 \end{array}\right].$$

For what values of k is there

(a) exactly one solution?

(b) infinitely many solutions?

18. Consider the system $\begin{cases} x + y + 2z = a \\ 2x + by + 4z = 1 \end{cases}$.

In each case below, determine all values of a and b which give the indicated number of solutions. If no values of a and b exist, explain why not.

(a) No solution.

(b) Exactly one solution.

(c) Infinitely many solutions.

1. (a) $\begin{array}{c} \begin{array}{cccc} & t & & \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right) \end{array}$ is a consistent system with infinite solutions since y lacks a leading 1 and so is a parameter

$x=0$
$y=t$ OR $\boxed{\vec{x}=(0,t,3)}$
$z=3$

(b) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$ Is inconsistent (no solution) since $0 \neq 1$.

(c) $\begin{array}{c} \begin{array}{ccccc} & t & & & \end{array} \\ \left(\begin{array}{cccc|c} 1 & 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$ Again x_2 is a parameter and so there are infinite solutions

$x_1=-5t$
$x_2=t$ OR $\boxed{\vec{x}=(-5t,t,0,0)}$
$x_3=0$
$x_4=0$

2(a) $\boxed{\left(\begin{array}{cccc|c} 1 & -2 & 1 & 2 & -3 \\ -2 & 4 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & -4 \end{array}\right)}$ (b) $\xrightarrow{R_2 \to R_2+2R_1} \left(\begin{array}{cccc|c} 1 & -2 & 1 & 2 & -3 \\ 0 & 0 & 2 & 3 & -5 \\ 0 & 0 & 1 & 2 & -4 \end{array}\right) R_2 \leftrightarrow R_3$

$\left(\begin{array}{cccc|c} 1 & -2 & 1 & 2 & -3 \\ 0 & 0 & 1 & 2 & -4 \\ 0 & 0 & 2 & 3 & -5 \end{array}\right) \begin{array}{l} R_1 \to R_1-R_2 \\ R_3 \to R_3-2R_2 \end{array} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & -1 & 3 \end{array}\right)$ $R_3 \to -R_3$ then $R_2 \to R_2-2R_3$

$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right)$ x_2 has no leading 1
Let $x_2=t$
leading 1's

(c) $\boxed{\begin{array}{l} x_1=1+2t \\ x_2=t \\ x_3=2 \\ x_4=-3 \end{array}}$

Check: $x_1-2x_2+x_3+2x_4=-3$
$(1+2t)-2t+2+2(-3)=-3$
$1+\cancel{2t}-\cancel{2t}+2-6=-3$
$-3=-3$ ✓

3.(a) $\left(\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ -1 & 1 & -2 & 0 \\ 3 & -3 & 5 & 1 \\ 1 & -1 & 0 & 2 \end{array}\right) \begin{array}{l} R_2 \to R_2+R_1 \\ R_3 \to R_3-3R_1 \\ R_4 \to R_4-R_1 \end{array} \left(\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & -3 & 3 \end{array}\right) \begin{array}{l} R_1 \to R_1-3R_2 \\ R_3 \to R_3+4R_2 \\ R_4 \to R_4+3R_2 \end{array}$

$\begin{array}{c} \begin{array}{cccc} & t & & \end{array} \\ \boxed{\left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)} \end{array}$ (b) y is a parameter (it lacks a leading 1) $\to y=t$

$\boxed{\begin{array}{l} x=2+t \\ y=t \\ z=-1 \end{array}}$

(c) $x=2+t=3 \to t=1$
$\therefore x=3, y=1, z=-1$
or $\boxed{(3,1,-1)}$

4. (a) There were $\boxed{5}$ equation b/c there are 5 rows in the matrix.
(b) There are $\boxed{4}$ variables b/c there are 4 columns in the coefficient matrix.
(c) The system has $\boxed{\text{no solutions}}$ b/c the 4th row $(0\ 0\ 0\ 0|1)$ says $0=1$ which is false.
(d) R has $\boxed{4}$ leading 1's b/c the first nonzero entry is a 1 in the first 4 rows.
(e) Rank of coefficient matrix is $\boxed{3}$ b/c it has 3 leading 1's (Note the 4th leading 1 is in the augment which is not a coefficient!)

5. Note homogeneous systems are guaranteed to be consistent (that is have one or more solutions) since no solution only occurs if $(0\ 0\ 0\ | k \to \text{where } k \neq 0)$. This can't happen since augment is strictly 0's. In this system we have only 2 rows but will have 4 variables → At least 2 of these variables will end up being parameters due to a lack of leading 1's. $\boxed{\text{Infinite solutions.}}$

6.(a) $A=\left[\begin{array}{cccc|c} 1 & 2 & 1 & -3 & 2 \\ 0 & 0 & 1 & 1 & 4 \\ 2 & 4 & 2 & -6 & 4 \end{array}\right] R_3 \to R_3-2 \cdot R_1 \left[\begin{array}{cccc|c} 1 & 2 & 1 & -3 & 2 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$R_1 \to R_1-1 \cdot R_2$ $\begin{array}{c} \begin{array}{ccccc} & s & & t & \end{array} \\ \left[\begin{array}{cccc|c} 1 & 2 & 0 & -4 & -2 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$ The system has infinitely many solutions

$\left.\begin{array}{l} x_1=-2-2s+4t \\ x_2=s \\ x_3=4-t \\ x_4=t \end{array}\right\} \to \boxed{\begin{array}{l} \text{vector form:} \\ (-2-2s+4t, s, 4-t, t) \end{array}}$

(b) $B=\left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & -5 \\ 0 & 1 & 0 & 1 & -4 \\ 2 & 4 & 4 & 6 & 5 \end{array}\right] R_3 \to R_3-2 \cdot R_2 \left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & -5 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 15 \end{array}\right]$

$\boxed{\text{This system has no solution } (0 \neq 15).}$

7.(a) $\left(\begin{array}{cccc|c} 2 & -4 & 2 & -2 & 2 \\ 2 & -4 & 3 & -3 & 2 \\ 4 & -8 & 3 & -4 & 5 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right) R_1 \to \tfrac{1}{2}R_1 \left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 1 \\ 2 & -4 & 3 & -3 & 2 \\ 4 & -8 & 3 & -4 & 5 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right) \begin{array}{l} R_2 \to R_2-2R_1 \\ R_3 \to R_3-4R_1 \end{array}$

$\left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right) \begin{array}{l} R_1 \to R_1-1R_2 \\ R_3 \to R_3+1R_2 \\ R_4 \to R_4+1R_2 \end{array} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) R_3 \to -R_3$ then $R_2 \to R_2+1R_3$

$\begin{array}{c} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{ccccc} & t & & & \end{array} \end{array}$ RREF $\begin{array}{l} x_1=1+2t \\ x_2=t \\ x_3=-1 \\ x_4=-1 \end{array}$ OR $\boxed{(1+2t, t, -1, -1)}$

(Check the solution if you have time.)

(b) Since the solution has only one parameter t, then, geometrically, the solution is a line.

(c) Since the augmented matrix has 3 leading 1's in RREF its $\boxed{\text{rank is } 3}$.

8. (a) $\left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 3 & 4 & -2 & 7 \end{array}\right) R_2 \to R_2-3R_1 \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -2 & 1 & -5 \end{array}\right)$

$R_2 \to -\tfrac{1}{2}R_2 \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & -\tfrac{1}{2} & \tfrac{5}{2} \end{array}\right) R_1 \to R_1-2R_2$ $\begin{array}{c} \begin{array}{cccc} & & t & \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & -\tfrac{1}{2} & \tfrac{5}{2} \end{array}\right) \end{array}$

leading 1's
No leading 1

x_3 is a parameter since its column lacks a leading 1 → Let $x_3=t$

$\boxed{\begin{array}{l} x_1=-1 \\ x_2=\tfrac{5}{2}+\tfrac{1}{2}t \\ x_3=t \end{array}}$

Check in $3x_1+4x_2-2x_3=7$
$3(-1)+4(\tfrac{5}{2}+\tfrac{1}{2}t)-2t=7$
$-3+10+\cancel{2t}-\cancel{2t}=7$
$7=7$ ✓

(b) $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 4 & -1 & 5 & 4 \\ 2 & 2 & -3 & 0 \end{array}\right) \begin{array}{l} R_2 \to R_2-4R_1 \\ R_3 \to R_3-2R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & -5 & 9 & -24 \\ 0 & 0 & -1 & -14 \end{array}\right)$

$R_2 \to -\tfrac{1}{5}R_2 \left(\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 1 & -\tfrac{9}{5} & \tfrac{24}{5} \\ 0 & 0 & -1 & -14 \end{array}\right) \begin{array}{l} R_1 \to R_1-R_2 \\ R_3 \to -R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & \tfrac{4}{5} & \tfrac{11}{5} \\ 0 & 1 & -\tfrac{9}{5} & \tfrac{24}{5} \\ 0 & 0 & 1 & 14 \end{array}\right)$

$\begin{array}{l} R_1 \to R_1-\tfrac{4}{5}R_3 \\ R_2 \to R_2+\tfrac{9}{5}R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\tfrac{45}{5}=-9 \\ 0 & 1 & 0 & \tfrac{150}{5}=30 \\ 0 & 0 & 1 & 14 \end{array}\right) \boxed{\begin{array}{l} x_1=-9 \\ x_2=30 \\ x_3=14 \end{array}}$

Check in $4x_1-x_2+5x_3=4$
$4(-9)-30+5(14)=4$
$-36-30+70=4$
$4=4$ ✓

8.(c)

$$\left(\begin{array}{ccc|c}1&1&-1&-1\\1&-1&5&5\\2&3&-5&-5\end{array}\right)\begin{array}{l}R_2\to R_2-R_1\\R_3\to R_3-2R_1\end{array}\left(\begin{array}{ccc|c}1&1&-1&-1\\0&-2&6&6\\0&1&-3&-3\end{array}\right)R_2\leftrightarrow R_3$$

$$\left(\begin{array}{ccc|c}1&1&-1&-1\\0&1&-3&-3\\0&-2&6&6\end{array}\right)\begin{array}{l}R_1\to R_1-R_2\\R_3\to R_3+2R_2\end{array}\begin{array}{c}\qquad\qquad t\\\left(\begin{array}{ccc|c}1&0&2&2\\0&1&-3&-3\\0&0&0&0\end{array}\right)\end{array}$$

z is a parameter b/c z lacks a leading 1
Let $z=t$

$$\boxed{\begin{array}{l}x=2-2t\\y=-3+3t\\z=t\end{array}}$$

Check in $2x+3y-5z=-5$

$2(2-2t)+3(-3+3t)-5t=-5$

$4-\cancel{4t}-9+\cancel{9t}-\cancel{5t}=-5$

$-5=-5$ ✓

8.(d)

$$\left(\begin{array}{ccc|c}1&2&3&1\\2&1&3&1\\1&1&2&1\end{array}\right)\begin{array}{l}R_2\to R_2-2R_1\\R_3\to R_3-R_1\end{array}\left(\begin{array}{ccc|c}1&2&3&1\\0&-3&-3&-1\\0&-1&-1&0\end{array}\right)$$

$R_2\leftrightarrow R_3$ then multiply new R_2 by -1

$$\left(\begin{array}{ccc|c}1&2&3&1\\0&1&1&0\\0&-3&-3&-1\end{array}\right)R_3\to R_3+3R_1\left(\begin{array}{ccc|c}1&2&3&1\\0&1&1&0\\0&0&0&-1\end{array}\right)$$

$\boxed{\text{No solution b/c } 0\neq-1}$

8.(e)

$$\left(\begin{array}{ccccc|c}1&1&1&1&1&7\\2&2&3&3&4&18\end{array}\right)R_2\to R_2-2R_1\left(\begin{array}{ccccc|c}1&1&1&1&1&7\\0&0&1&1&2&4\end{array}\right)R_1\to R_1-R_2$$

$$\begin{array}{c}\quad r\qquad s\quad t\\\left(\begin{array}{ccccc|c}1&1&0&0&-1&3\\0&0&1&1&2&4\end{array}\right)\end{array}$$

leading 1's

x_2, x_4 and x_5 are parameters (they lack leading 1's)

Let $x_2=r$, $x_4=s$, $x_5=t$

$$\boxed{\begin{array}{l}x_1=3-r+t\\x_2=r\\x_3=4-s-2t\\x_4=s\\x_5=t\end{array}}$$

Vector Form of answer required:

$\boxed{\vec{x}=(3-r+t,\ r,\ 4-s-2t,\ s,\ t)}$

Remember to check by subbing solution into an original equation.

8.(f)

$$\left(\begin{array}{cccc|c}2&2&-1&0&0\\-1&-1&2&-3&0\\1&1&-2&0&0\\0&0&1&1&0\end{array}\right)R_1\leftrightarrow R_3\left(\begin{array}{cccc|c}1&1&-2&0&0\\-1&-1&2&-3&0\\2&2&-1&0&0\\0&0&1&1&0\end{array}\right)$$

$$\begin{array}{l}R_2\to R_2+R_1\\R_3\to R_3-2R_1\end{array}\left(\begin{array}{cccc|c}1&1&-2&0&0\\0&0&0&-3&0\\0&0&3&0&0\\0&0&1&1&0\end{array}\right)R_2\leftrightarrow R_4\left(\begin{array}{cccc|c}1&1&-2&0&0\\0&0&1&1&0\\0&0&3&0&0\\0&0&0&-3&0\end{array}\right)\begin{array}{l}R_1\to R_1+2R_2\\R_3\to R_3-3R_2\end{array}$$

$$\left(\begin{array}{cccc|c}1&1&0&2&0\\0&0&1&1&0\\0&0&0&-3&0\\0&0&0&-3&0\end{array}\right)R_3\to-\tfrac{1}{3}R_3\left(\begin{array}{cccc|c}1&1&0&2&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&-3&0\end{array}\right)\begin{array}{l}R_1\to R_1-2R_3\\R_2\to R_2-R_3\\R_4\to R_4+3R_3\end{array}$$

$$\begin{array}{c}\quad t\qquad\qquad\\\left(\begin{array}{cccc|c}1&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0\end{array}\right)\end{array}$$

leading 1's

x_2 lacks a leading 1 → parameter

Let $x_2=t$

$$\boxed{\begin{array}{l}x_1=-t\\x_2=t\\x_3=0\\x_4=0\end{array}}$$

8.(g)

$$\left(\begin{array}{cccc|c}1&2&0&2&5\\2&4&2&4&12\\-1&-2&3&-2&-2\\0&1&0&-3&0\end{array}\right)\begin{array}{l}R_2\to R_2-2R_1\\R_3\to R_3+R_1\end{array}\left(\begin{array}{cccc|c}1&2&0&2&5\\0&0&2&0&2\\0&0&3&0&3\\0&1&0&-3&0\end{array}\right)R_2\leftrightarrow R_4$$

$$\left(\begin{array}{cccc|c}1&2&0&2&5\\0&1&0&-3&0\\0&0&3&0&3\\0&0&2&0&2\end{array}\right)R_1\to R_1-2R_2\left(\begin{array}{cccc|c}1&0&0&8&5\\0&1&0&-3&0\\0&0&3&0&3\\0&0&2&0&2\end{array}\right)R_3\to\tfrac{1}{3}R_3$$

$$\left(\begin{array}{cccc|c}1&0&0&8&5\\0&1&0&-3&0\\0&0&1&0&1\\0&0&2&0&2\end{array}\right)R_4\to R_4-2R_3\begin{array}{c}\qquad\qquad t\quad\\\left(\begin{array}{cccc|c}1&0&0&8&5\\0&1&0&-3&0\\0&0&1&0&1\\0&0&0&0&0\end{array}\right)\end{array}$$

leading 1's

x_4 lacks a leading 1 and so is a parameter $x_4=t$

$\boxed{x_1=5-8t,\ x_2=3t,\ x_3=1,\ x_4=t}$

8.(h)

$$\left(\begin{array}{ccc|c}1&1&-1&1\\2&1&2&5\\4&3&0&7\end{array}\right)\begin{array}{l}R_2\to R_2-2R_1\\R_3\to R_3-4R_1\end{array}\left(\begin{array}{ccc|c}1&1&-1&1\\0&-1&4&3\\0&-1&4&3\end{array}\right)R_2\to-R_2$$

$$\left(\begin{array}{ccc|c}1&1&-1&1\\0&1&-4&-3\\0&-1&4&3\end{array}\right)\begin{array}{l}R_1\to R_1-R_2\\R_3\to R_3+R_2\end{array}\begin{array}{c}\qquad\quad t\\\boxed{\left(\begin{array}{ccc|c}1&0&3&4\\0&1&-4&-3\\0&0&0&0\end{array}\right)}\end{array}$$

$z=t$

z is a parameter (no leading 1 in its column)

$$\boxed{\begin{array}{l}x=4-3t\\y=-3+4t\\z=t\end{array}}$$

Check in $2x+y+2z=5$

$2(4-3t)+(-3+4t)+2t=5$

$8-\cancel{6t}-3+\cancel{4t}+\cancel{2t}=5$

$5=5$ ✓

9.(a)

$$\left(\begin{array}{cccc|c}1&2&0&2&4\\2&4&2&5&2\\3&6&2&7&6\end{array}\right)\begin{array}{l}R_2\to R_2-2R_1\\R_3\to R_3-3R_1\end{array}\left(\begin{array}{cccc|c}1&2&0&2&4\\0&0&2&1&-6\\0&0&2&1&-6\end{array}\right)\begin{array}{l}R_2\to\frac{1}{2}R_2\\R_3\to R_3-R_2\end{array}$$

$$\begin{array}{c}\quad s\qquad t\quad\\\left(\begin{array}{cccc|c}1&2&0&2&4\\0&0&1&\frac{1}{2}&-3\\0&0&0&0&0\end{array}\right)\end{array}$$

leading 1's

x_2 & x_4 are parameters (they lack leading 1's) $x_2=s$, $x_4=t$

$$\boxed{\begin{array}{l}x_1=4-2s-2t\\x_2=s\\x_3=-3-\frac{1}{2}t\\x_4=t\end{array}}$$

9.(b)

$$\left(\begin{array}{ccccc|c}1&1&1&1&1&2\\1&1&1&2&2&3\\1&1&1&2&3&2\end{array}\right)\begin{array}{l}R_2\to R_2-R_1\\R_3\to R_3-R_1\end{array}\left(\begin{array}{ccccc|c}1&1&1&1&1&2\\0&0&0&1&1&1\\0&0&0&1&2&0\end{array}\right)\begin{array}{l}R_1\to R_1-R_2\\R_3\to R_3-R_2\end{array}$$

$$\left(\begin{array}{ccccc|c}1&1&1&0&0&1\\0&0&0&1&1&1\\0&0&0&0&1&-1\end{array}\right)R_2\to R_2-R_3\begin{array}{c}\quad s\ t\qquad\qquad\\\left(\begin{array}{ccccc|c}1&1&1&0&0&1\\0&0&0&1&0&2\\0&0&0&0&1&-1\end{array}\right)\end{array}$$

leading 1's

x_2 and x_3 are parameters (they lack leading 1's)

Let $x_2=s$ and $x_3=t$

$$\boxed{\begin{array}{l}x_1=1-s-t\\x_2=s\\x_3=t\\x_4=2\\x_5=-1\end{array}}$$

9.(c) Get x's, y's and z's on left in that order, #'s on right:

$$\begin{aligned}x+y+z&=6\\x-y+z&=2\\x+y-z&=4\end{aligned}$$

$$\left(\begin{array}{ccc|c}1&1&1&6\\1&-1&1&2\\1&1&-1&4\end{array}\right)\begin{array}{l}R_2\to R_2-R_1\\R_3\to R_3-R_1\end{array}\left(\begin{array}{ccc|c}1&1&1&6\\0&-2&0&-4\\0&0&-2&-2\end{array}\right)R_2\to-\tfrac{1}{2}R_2$$

$$\left(\begin{array}{ccc|c}1&1&1&6\\0&1&0&2\\0&0&-2&-2\end{array}\right)\begin{array}{l}R_1\to R_1-R_2\\R_3\to-\frac{1}{2}R_3\end{array}\left(\begin{array}{ccc|c}1&0&1&4\\0&1&0&2\\0&0&1&1\end{array}\right)R_1\to R_1-R_3$$

$$\left(\begin{array}{ccc|c}1&0&0&3\\0&1&0&2\\0&0&1&1\end{array}\right)\quad\boxed{x=3,\ y=2,\ z=1}$$

9.(d)
$$\begin{aligned} x - y \quad &= 5 \\ 3x + y - z &= 3 \\ -x + 2y + z &= -8 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 5 \\ 3 & 1 & -1 & 3 \\ -1 & 2 & 1 & -8 \end{array}\right) \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + R_1 \end{array} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 5 \\ 0 & 4 & -1 & -12 \\ 0 & 1 & 1 & -3 \end{array}\right) R_2 \leftrightarrow R_3$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 4 & -1 & -12 \end{array}\right) \begin{array}{l} R_1 \to R_1 + R_2 \\ R_3 \to R_3 - 4R_2 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & -5 & 0 \end{array}\right) R_3 \to -\tfrac{1}{5}R_3 \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\begin{array}{l} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{array}\right) \qquad \boxed{x = 2,\ y = -3,\ z = 0}$$

Check (using 2nd equation): $3x + y - z = 3 \to 3(2) + (-3) - 0 = 3$ ✓

9.(e)
$$\left(\begin{array}{cccc|c} 2 & 2 & 2 & 2 & 2 \\ 1 & 0 & 1 & 0 & 3 \\ 2 & 1 & 2 & 1 & 4 \end{array}\right) \quad R_1 \leftrightarrow R_2$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 1 & 2 & 1 & 4 \end{array}\right) \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 2 & -4 \\ 0 & 1 & 0 & 1 & -2 \end{array}\right) R_2 \leftrightarrow R_3 \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 2 & 0 & 2 & -4 \end{array}\right)$$

$$R_3 \to R_3 - 2R_2 \begin{array}{c} \begin{array}{cccccc} & & s & t & & \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

x_3 and x_4 are parameters b/c their columns lack leading 1's.

$$\boxed{\begin{aligned} x_1 &= 3 - s \\ x_2 &= -2 - t \\ x_3 &= s \\ x_4 &= t \end{aligned}} \quad \text{OR} \quad \boxed{\overline{x} = (3 - s, -2 - t, s, t)}$$

Check: $2x_1 + x_2 + 2x_3 + x_4 = 4$
$2(3 - s) + (-2 - t) + 2s + t = 4$
$6 - \cancel{2s} - 2 - \cancel{t} + \cancel{2s} + \cancel{t} = 4$
$4 = 4$ ✓

9.(f)
$$\left(\begin{array}{ccc|c} 0 & 3 & 3 & -3 \\ 1 & -2 & -3 & 2 \\ -1 & 4 & 6 & 8 \end{array}\right) R_1 \leftrightarrow R_2 \left(\begin{array}{ccc|c} 1 & -2 & -3 & 2 \\ 0 & 3 & 3 & -3 \\ -1 & 4 & 6 & 8 \end{array}\right) \begin{array}{l} R_2 \to \tfrac{1}{3}R_2 \\ R_3 \to R_3 + 1 \cdot R_1 \end{array} \left(\begin{array}{ccc|c} 1 & -2 & -3 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 3 & 10 \end{array}\right)$$

$$\begin{array}{l} R_1 \to R_1 + 2 \cdot R_2 \\ R_3 \to R_3 - 2 \cdot R_2 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 12 \end{array}\right) \begin{array}{l} R_1 \to R_1 + 1 \cdot R_3 \\ R_2 \to R_2 - 1 \cdot R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & -13 \\ 0 & 0 & 1 & 12 \end{array}\right)$$

Solution: $\boxed{x_1 = 12,\ x_2 = -13,\ x_3 = 12}$ OR $\boxed{(12, -13, 12)}$

9.(g)
$$\left(\begin{array}{ccc|c} 3 & 3 & 6 & 3 \\ 2 & 1 & -1 & 3 \\ 0 & -1 & -5 & 1 \\ -5 & -2 & 5 & -8 \end{array}\right) \quad R_1 \to \tfrac{1}{3}R_1 \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 2 & 1 & -1 & 3 \\ 0 & -1 & -5 & 1 \\ -5 & -2 & 5 & -8 \end{array}\right) \begin{array}{l} R_2 \to R_2 - 2R_1 \\ \\ R_4 \to R_4 + 5R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & +1 & +5 & -1 \\ 0 & -1 & -5 & 1 \\ 0 & 3 & 15 & -3 \end{array}\right) R_2 \to -R_2 \text{ then } \begin{array}{l} R_1 \to R_1 - 1R_2 \\ R_3 \to R_3 + 1R_2 \\ R_4 \to R_4 - 3R_2 \end{array} \begin{array}{c} \begin{array}{cccc} & & t & \end{array} \\ \boxed{\left(\begin{array}{ccc|c} 1 & 0 & -3 & 2 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)} \end{array} \quad \begin{array}{c} \text{Done} \\ \text{RREF} \end{array}$$

$$\begin{aligned} x_1 &= 2 + 3t \\ x_2 &= -1 - 5t \\ x_3 &= t \end{aligned} \quad \text{OR} \quad \boxed{(2 + 3t, -1 - 5t, t)}$$

9.(h)
$$\left(\begin{array}{cccc|c} 1 & 2 & -5 & 1 & 0 \\ 2 & -2 & 8 & 8 & 6 \\ 3 & 2 & -3 & 7 & 4 \end{array}\right) \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array} \left(\begin{array}{cccc|c} 1 & 2 & -5 & 1 & 0 \\ 0 & -6 & 18 & 6 & 6 \\ 0 & -4 & 12 & 4 & 4 \end{array}\right) R_2 \to -\tfrac{1}{6}R_2$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -5 & 1 & 0 \\ 0 & 1 & -3 & -1 & -1 \\ 0 & -4 & 12 & 4 & 4 \end{array}\right) \begin{array}{l} R_1 \to R_1 - 2R_2 \\ R_3 \to R_3 + 4R_2 \end{array} \begin{array}{c} \begin{array}{ccccc} & & s & t & \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 1 & 3 & 2 \\ 0 & 1 & -3 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

x_3 & x_4 are parameters (lack leading 1's)

$$\begin{aligned} x_1 &= 2 - s - 3t \\ x_2 &= -1 + 3s + t \\ x_3 &= s \\ x_4 &= t \end{aligned} \quad \text{OR} \quad \boxed{\overline{x} = (2 - s - 3t, -1 + 3s + t, s, t)}$$

Check in $3x_1 + 2x_2 - 3x_3 + 7x_4 = 4$ (or the others)
$3(2 - s - 3t) + 2(-1 + 3s + t) - 3s + 7t = 4$
$6 - 3s - 9t - 2 + 6s + 2t - 3s + 7t = 4$
$4 = 4$ ✓

9.(i)
$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 2 \\ -2 & 0 & 1 & 0 & -4 \\ 1 & -1 & -2 & 2 & -1 \end{array}\right) \begin{array}{l} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - R_1 \end{array} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 2 \\ 0 & +2 & -1 & 2 & 0 \\ 0 & -2 & -1 & 1 & -3 \end{array}\right) R_2 \to \tfrac{1}{2}R_2$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 2 \\ 0 & 1 & -\tfrac{1}{2} & 1 & 0 \\ 0 & -2 & -1 & 1 & -3 \end{array}\right) \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 + 2R_2 \end{array} \left(\begin{array}{cccc|c} 1 & 0 & -\tfrac{1}{2} & 0 & 2 \\ 0 & 1 & -\tfrac{1}{2} & 1 & 0 \\ 0 & 0 & -2 & 3 & -3 \end{array}\right) R_3 \to -\tfrac{1}{2}R_3$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -\tfrac{1}{2} & 0 & 2 \\ 0 & 1 & -\tfrac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & -\tfrac{3}{2} & \tfrac{3}{2} \end{array}\right) \begin{array}{l} R_1 \to R_1 + \tfrac{1}{2}R_3 \\ R_2 \to R_2 + \tfrac{1}{2}R_3 \end{array} \begin{array}{c} \begin{array}{ccccc} & & & t & \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & -\tfrac{3}{4} & \tfrac{11}{4} \\ 0 & 1 & 0 & \tfrac{1}{4} & \tfrac{3}{4} \\ 0 & 0 & 1 & -\tfrac{3}{2} & \tfrac{3}{2} \end{array}\right) \end{array} \ \text{Done}$$

$$\begin{aligned} x_1 &= \tfrac{11}{4} + \tfrac{3}{4}t \\ x_2 &= \tfrac{3}{4} - \tfrac{1}{4}t \\ x_3 &= \tfrac{3}{2} + \tfrac{3}{2}t \end{aligned} \quad \text{OR} \quad \boxed{\overline{x} = \left(\tfrac{11}{4} + \tfrac{3}{4}t, \tfrac{3}{4} - \tfrac{1}{4}t, \tfrac{3}{2} + \tfrac{3}{2}t, t\right)}$$

9.(j)
$$\left(\begin{array}{ccccc|c} 1 & 3 & -1 & 1 & -2 & 8 \\ 1 & 6 & -4 & 2 & -1 & 14 \\ 2 & 8 & -4 & 3 & -4 & 21 \end{array}\right) \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array} \left(\begin{array}{ccccc|c} 1 & 3 & -1 & 1 & -2 & 8 \\ 0 & 3 & -3 & 1 & 1 & 6 \\ 0 & 2 & -2 & 1 & 0 & 5 \end{array}\right)$$

$$R_2 \to R_2 - R_3 \left(\begin{array}{ccccc|c} 1 & 3 & -1 & 1 & -2 & 8 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 2 & -2 & 1 & 0 & 5 \end{array}\right) \begin{array}{l} R_1 \to R_1 - 3R_2 \\ R_3 \to R_3 - 2R_2 \end{array} \left(\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & -5 & 5 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 3 \end{array}\right)$$

$$R_1 \to R_1 - R_3 \begin{array}{c} \begin{array}{cccccc} & & s & & t & \end{array} \\ \left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -3 & 2 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 3 \end{array}\right) \end{array} \quad \text{is row reduced echelon form.}$$

$$\boxed{\begin{aligned} x_1 &= 2 - 2s + 3t \\ x_2 &= 1 + s - t \\ x_3 &= s \\ x_4 &= 3 + 2t \\ x_5 &= t \end{aligned}}$$

9.(k)
$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{array}\right) \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right)$$

$$\begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - R_2 \end{array} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right) R_2 \to R_2 - R_3 \begin{array}{c} \boxed{\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right)} \\ \begin{array}{cccccc} & s & t & & & \end{array} \end{array}$$

$$\boxed{\begin{aligned} x_1 &= 1 - s - t \\ x_2 &= s \\ x_3 &= t \\ x_4 &= 2 \\ x_5 &= -1 \end{aligned}}$$

9.(l)

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 3 \\ -2 & 1 & 1 & -4 & -7 \\ 0 & 0 & 1 & -1 & 1 \\ -2 & 1 & -1 & -2 & -9 \end{array}\right) \begin{array}{l} R_2 \to R_2 + 2R_1 \\ \\ R_4 \to R_4 + 2R_1 \end{array} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 2 & -3 \end{array}\right) R_4 \to R_4 - R_2$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 2 & -2 \end{array}\right) \begin{array}{l} R_2 \to R_2 - R_3 \\ \\ R_4 \to R_4 + 2R_3 \end{array} \begin{array}{c} \qquad\qquad\quad t \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array} \begin{array}{l} x_1 = 3 - 2t \\ x_2 = -2 - t \\ x_3 = 1 + t \\ x_4 = t \end{array}$$

ie $\boxed{(3-2t, -2-t, 1+t, t)}$ where t is any real number

9.(m) Be careful to line the columns up!

$$\begin{array}{c} x_1\ \ x_2\ \ x_3\ \ x_4\ \ x_5 \qquad \\ \left(\begin{array}{ccccc|c} 7 & 3 & 1 & 0 & 0 & 0 \\ 7 & 3 & 0 & 1 & 1 & 1 \end{array}\right) \end{array}$$

Change the order of the unknowns (You can do that at the start but never again!)

$$\begin{array}{c} \qquad\quad r\ \ \ s\ \ \ t \qquad \\ x_3\ x_4\ x_5\ x_1\ x_2 \qquad \\ \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 7 & 3 & 0 \\ 0 & 1 & 1 & 7 & 3 & 1 \end{array}\right) \end{array} \begin{array}{l} x_3 = -7s - 3t \\ x_4 = 1 - r - 7s - 3t \\ x_5 = r \\ x_1 = s \\ x_2 = t \end{array}$$

Now, give the solution vector in standard order $(x_1, x_2, x_3, x_4, x_5)$

$\therefore$ $\boxed{(s, t, -7s-3t, 1-r-7s-3t, r)}$ is the solution in vector form.

<u>OR</u> Conventional method:

$$\left(\begin{array}{ccccc|c} 7 & 3 & 1 & 0 & 0 & 0 \\ 7 & 3 & 0 & 1 & 1 & 1 \end{array}\right) R_1 \to \tfrac{1}{7} R_1 \left(\begin{array}{ccccc|c} 1 & 3/7 & 1/7 & 0 & 0 & 0 \\ 7 & 3 & 0 & 1 & 1 & 1 \end{array}\right) R_2 \to R_2 - 7R_1$$

$$\left(\begin{array}{ccccc|c} 1 & 3/7 & 1/7 & 0 & 0 & 0 \\ 0 & 0 & \oplus 1 & \ominus 1 & \ominus 1 & \ominus 1 \end{array}\right) R_2 \to -R_2 \text{ then } R_1 \to R_1 - \tfrac{1}{7} R_2 \begin{array}{c} \quad r \qquad s\ \ \ t \quad \\ \left(\begin{array}{ccccc|c} 1 & 3/7 & 0 & 1/7 & 1/7 & 1/7 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array}\right) \end{array}$$

Giving us: $\boxed{\left(\tfrac{1}{7} - \tfrac{3}{7} r - \tfrac{1}{7} s - \tfrac{1}{7} t, r, -1+s+t, s, t\right)}$

[Note: Even though both solution are very different, both are correct.]

9.(n) If you use recipricals to create leading 1's in this problem, you will be dealing with lots of fractions. It would be correct but messy. Try this way instead:

$$\left(\begin{array}{ccc|c} -40 & 16 & 9 & 1 \\ 13 & -5 & -3 & -2 \\ 5 & -2 & -1 & -1 \end{array}\right) R_1 \to R_1 + 3R_2 \left(\begin{array}{ccc|c} -1 & 1 & 0 & -5 \\ 13 & -5 & -3 & -2 \\ 5 & -2 & -1 & -1 \end{array}\right) R_1 \to -R_1$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 5 \\ 13 & -5 & -3 & -2 \\ 5 & -2 & -1 & -1 \end{array}\right) \begin{array}{l} R_2 \to R_2 - 13R_1 \\ R_3 \to R_3 - 5R_1 \end{array} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 5 \\ 0 & 8 & -3 & -67 \\ 0 & 3 & -1 & -26 \end{array}\right) R_2 \to R_2 - 3R_3$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 5 \\ 0 & -1 & 0 & 11 \\ 0 & 3 & -1 & -26 \end{array}\right) R_2 \to -R_2 \left(\begin{array}{ccc|c} 1 & -1 & 0 & 5 \\ 0 & 1 & 0 & -11 \\ 0 & 3 & -1 & -26 \end{array}\right) \begin{array}{l} R_1 \to R_1 + R_2 \\ R_3 \to R_3 - 3R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & -11 \\ 0 & 0 & -1 & 7 \end{array}\right) R_3 \to -R_3 \left(\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & -11 \\ 0 & 0 & 1 & -7 \end{array}\right) \boxed{\begin{array}{l} x = -6 \\ y = -11 \\ z = -7 \end{array}}$$

10.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & K+1 & 0 & 0 \\ 0 & 0 & K & 5 \end{array}\right)$$

$\to$ Note this row would be $(0\ 0\ 0 \mid 5)$ if $K=0$, causing an inconsistent system and so no solution.

$\to$ If $K=-1$ this row would be $(0\ 0\ 0 \mid 0)$ and the matrix would be $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 5 \end{array}\right)$

causing y to be a parameter and so infinite solutions.

If $K \neq -1$ <u>and</u> $K \neq 0$ we would have nonzero numbers down the main diagonal. These #'s could be converted to leading 1's in every column and so a unique solution.

(a) Unique solution if $\boxed{K \neq 0 \text{ and } K \neq -1}$.

(b) Infinitely many solutions if $\boxed{K=-1}$.

(c) No solution if $\boxed{K=0}$.

11.

$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ 1/2 & -1 & c \end{array}\right) R_2 \to R_2 - \tfrac{1}{2} R_1 \left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & c - 1/2 \end{array}\right)$$

A system has no solution if and only if its row-reduced matrix has $(0\ 0 \mid \# \neq 0)$

$\therefore$ No solutions if $c - 1/2 \neq 0 \to \boxed{c \neq \tfrac{1}{2}}$

12.

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & p & q & 1 \end{array}\right) R_3 \to R_3 - pR_2 \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & q-p & 1-p \end{array}\right)$$

We know a system is inconsistent (ie no solution if we get $(0\ 0\ 0 \mid \# \neq 0)$

$\therefore$ If $q - p = 0$ (ie $q = p$) <u>and</u> $1 - p \neq 0$ (ie $p \neq 1$)

This system will be <u>inconsistent</u> (Answers **(c)**)

If we get $(0\ 0\ 0 \mid 0)$ however, the system would be consistent with infinite solutions (b/c z column lacks a leading 1 and so z is a parameter)

$\therefore$ If $q - p = 0$ (ie $q = p$) <u>and</u> $1 - p = 0$ (ie $p = 1$)

ie $q = p = 1$, we have infinite solutions

(Answers **(b)**)

If $q - p \neq 0$ we would have a nonzero value in 3rd row, 3rd column. That nonzero would then be able to become a leading 1 by multiplying by the reciprical, causing a leading 1 in every column $\to$ unique solution if $q - p \neq 0$ ie ($q \neq p$)

(The $1-p$ value would then be irrelevant)

In summary:

(a) unique solution if $q \neq p$
(b) infinite solutions if $q = p = 1$
(c) no solution if $q = p$ and $p \neq 1$

13. (a) $\boxed{\text{4 equations and 3 variables}}$ since there are 4 rows and 3 columns in the coefficient matrix.

(b) To have a unique solution we must have a leading 1 in every column. Columns 1 and 2 have a leading 1. As long as $a-2 \neq 0$ it will be a # which can be made into a leading 1 $\rightarrow$ $\boxed{\text{unique solution if } a-2\neq 0 \text{ ie } a\neq 2 \text{, b is irrelevant}}$

(c) No solution if $(0\ 0\ 0\,|\,\#\neq 0)$ ie $a-2=0 \rightarrow \boxed{a=2 \text{ and } b\neq 0}$

(d) Infinite solutions if $(0\ 0\ 0\,|\,0)$ $\rightarrow a-2=0$, $b=0$ since the system is consistent but column 3 lacks a leading 1 $\rightarrow$ making it variable a parameter.

$\therefore$ $\boxed{a=2,\ b=0 \text{ makes infinite solutions with one parameter.}}$

14. (a) $\boxed{\text{4 equations and 3 variables}}$ b/c there are 4 rows and 3 columns in the coefficient matrix.

(b) No solution if $(0\ 0\ 0\,|\,\#\neq 0)$. Therefore $b+1\neq 0 \rightarrow \boxed{b\neq -1}$ causes no solution, a is irrelevant.

(c) To have a unique solution, the system must first be consistent $\rightarrow b+1=0 \rightarrow b=-1$ causing $(0\ 0\ 0\,|\,0)$. Next, we must be able to get a leading 1 in every column $\rightarrow$ columns 1 and 2 already have a leading 1. As long as $a\neq 0$ we can make a leading 1 in column 3

$\therefore$ $\boxed{a\neq 0,\ b=-1}$ causes a unique solution.

15. (a) $\boxed{\text{3 equations and 3 variables}}$ since there are 3 rows and 3 columns in the coefficient matrix.

(b)
$$\left(\begin{array}{ccc|c} 2 & -3 & 5 & a \\ 1 & -1 & 2 & 0 \\ -4 & 6 & -10 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 2 & -3 & 5 & a \\ -4 & 6 & -10 & 1 \end{array}\right) \begin{matrix} R_2 \to R_2-2R_1 \\ R_3 \to R_3+4R_1 \end{matrix}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & -1 & 1 & a \\ 0 & 2 & -2 & 1 \end{array}\right) \xrightarrow{R_2 \to -R_2} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & -a \\ 0 & 2 & -2 & 1 \end{array}\right) R_3 \to R_3 - 2R_2$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & -a \\ 0 & 0 & 0 & 1+2a \end{array}\right)$$

$\rightarrow$ inconsistent (no solution) if $1+2a\neq 0$ i.e. $2a\neq -1$ $\boxed{a\neq -\tfrac{1}{2}}$

(c) If $1+2a=0 \rightarrow \boxed{a=-\tfrac{1}{2}}$ the system will have infinite solutions since x_3 is a parameter (No leading 1 in the 3rd column.)

16.
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & x & y \end{array}\right)$$
$\rightarrow$ if $(0\ 0\ 0\,|\,\#\neq 0)$ then ~~no~~ no solution

(a) $\boxed{x=0,\ y\neq 0}$ causes no solution

(b) If $\boxed{x\neq 0}$ there will be a unique solution since we can get leading 1's in all 3 columns (y is irrelevant)

(c) If $\boxed{x=0,\ y=0}$ there will be infinite solutions with x_3 as a parameter.

17.
$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -5 \\ 2 & -5 & k & 3k+5 \end{array}\right) \begin{matrix} R_2 \to R_2+R_1 \\ R_3 \to R_3-2R_1 \end{matrix} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 4 \\ 0 & -1 & k-6 & 3k-13 \end{array}\right)$$

$$\begin{matrix} R_1 \to R_1+2R_2 \\ R_3 \to R_3+R_2 \end{matrix} \left(\begin{array}{ccc|c} 1 & 0 & 9 & 17 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & k-3 & 3k-9 \end{array}\right)$$

STOP here. This system is inconsistent if $k-3=0$ and $3k-9\neq 0$ b/c that would cause $(0\ 0\ 0\,|\,\text{nonzero})$

But $k-3=0$ means $k=3$. If $k=3$ we get $(0\ 0\ 0\,|\,0)$ in the 3rd row.

Thus $k=3$ causes $\left(\begin{array}{ccc|c} 1 & 0 & 9 & 17 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$ which means infinite solutions b/c the 3rd column lacks a leading 1 and so that variable is a parameter.

If $k\neq 3$ we will have a nonzero entry in 3rd row, 3rd column $(0\ 0 \text{ Here}\,|\ )$

Any nonzero entry can be made into a leading 1 (by multiplying by the reciprocal). And so, $k\neq 3$ means we can achieve leading 1's in all the rows thus getting a unique solution.

(a) $k\neq 3$ gives exactly one solution

(b) $k=3$ gives infinitely many solutions.

18.
$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 2 & b & 4 & 1 \end{array}\right) R_2 \to R_2 - 2R_1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & b-2 & 0 & 1-2a \end{array}\right) \rightarrow \text{3 Cases}$$

Case 1: $(0\ \underset{b-2}{0}\ 0\,|\,\underset{1-2a}{\text{nonzero}})$

$b-2=0 \rightarrow b=2$ and $1-2a\neq 0 \rightarrow 1\neq 2a$, $a\neq \tfrac{1}{2}$

(a) $\boxed{\text{No solution if } b=2 \text{ and } a\neq \tfrac{1}{2}.}$

(b) This system could never have a unique solution. The best it could achieve is Rank = 2, but there are 3 variables.

(c) Case 2 and Case 3 will produce infinite solutions.

Case 2: $(0\ \underset{b-2}{0}\ 0\,|\,\underset{1-2a}{0}) \rightarrow b=2,\ a=\tfrac{1}{2}$

Case 3: $(0\ \underset{b-2}{\text{nonzero}}\ 0\,|\,\text{irrelevant}) \rightarrow b-2\neq 0 \rightarrow b\neq 2$

$\boxed{\text{Infinite solutions if } b\neq 2 \text{ OR if } b=2 \text{ and } a=\tfrac{1}{2}.}$