Lesson 15: Equations of Lines in Euclidean 3-Space

Equations of Lines:

To get the equation of a line in Euclidean 3-Space (R^3), we need a **point on the line**, $\mathbf{p} = (x_0, y_0, z_0)$, and a **vector parallel to the line**, $\mathbf{v} = (a, b, c)$, then we can state the equation of the line.

☑ Parametric equations: $x = x_0 + at,\ \ y = y_0 + bt,\ \ z = z_0 + ct$

☑ Vector form: $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ i.e. $(x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

1. Find the vector form of the line passing through the point (4, 1, 2) and parallel to the vector $3\mathbf{i} + 7\mathbf{j} + 8\mathbf{k}$.

2. Find the parametric equations of the line passing through the point (4, 1, 7) and parallel to the vector $5\mathbf{i} + 10\mathbf{j} + 12\mathbf{k}$.

3. Let $A = (1, -2, 3)$ and $B = (2, -3, 1)$ be points in R^3, find the parametric equations of the line passing through A and B.

4. Find the point of intersection (if any) of the lines: $x = 1 + t,\ y = 2 - 3t,\ z = 4 + 2t$ and $x = 9 + 7s,\ y = 4 + 5s,\ z = 9 + 3s$.

5. Find the line of intersection of the two planes: $x + 2y - 3z = 5$ and $x + 3y + z = 9$.

1. We are given a point on the line $\bar{p} = (4, 1, 2)$ and a vector parallel to the line $\bar{v} = 3\hat{i} + 7\hat{j} + 8\hat{k} = (3, 7, 8)$

Perfect! Vector form is $\bar{x} = \bar{p} + t\bar{v}$

$\therefore \boxed{(x, y, z) = (4, 1, 2) + t(3, 7, 8)}$

2. Given $\bar{p} = (4, 1, 7)$ and $\bar{v} = 5\hat{i} + 10\hat{j} + 12\hat{k} = (5, 10, 12)$ parallel to the line.

Perfect! To get parametric equations it is easier to write in columns

$$\begin{matrix} x = \\ y = \\ z = \end{matrix} \; \bar{p} + t\bar{v}$$

→ put $t\bar{v}$ in a column

Put $\bar{p}$ in a column

$$\boxed{\begin{aligned} x &= 4 + 5t \\ y &= 1 + 10t \\ z &= 7 + 12t \end{aligned}}$$

$\therefore \boxed{x = 4 + 5t,\ y = 1 + 10t,\ z = 7 + 12t}$

Always use the <u>parametric equation</u> of a line if you are needing the line to solve any other problem.

A line is an infinite number of points strung together. That's why there is a parameter "t" in the equation of a line. We can generate points on the line by selecting values for t.

<u>eg.</u> For the line in problem 1 above $(x, y, z) = (4, 1, 2) + t(3, 7, 8)$ we can let "$t = 1$" to get the point $(7, 8, 10)$ on the line. By subbing in "$t = -3$", we discover $(-5, -20, -22)$ is on the line.

<u>eg.</u> Is the point $(1, 2, 3)$ on the line from problem 2?

$$\underbrace{x = 4 + 5t}_{=1}, \quad \underbrace{y = 1 + 10t}_{=2}, \quad \underbrace{z = 7 + 12t}_{=3}$$

$$4 + 5t = 1 \qquad\qquad 1 + 10t = 2$$
$$5t = -3 \qquad\qquad 10t = 1$$
$$t = -3/5 \qquad\qquad t = 1/10 \quad \underline{\text{NO!}}$$

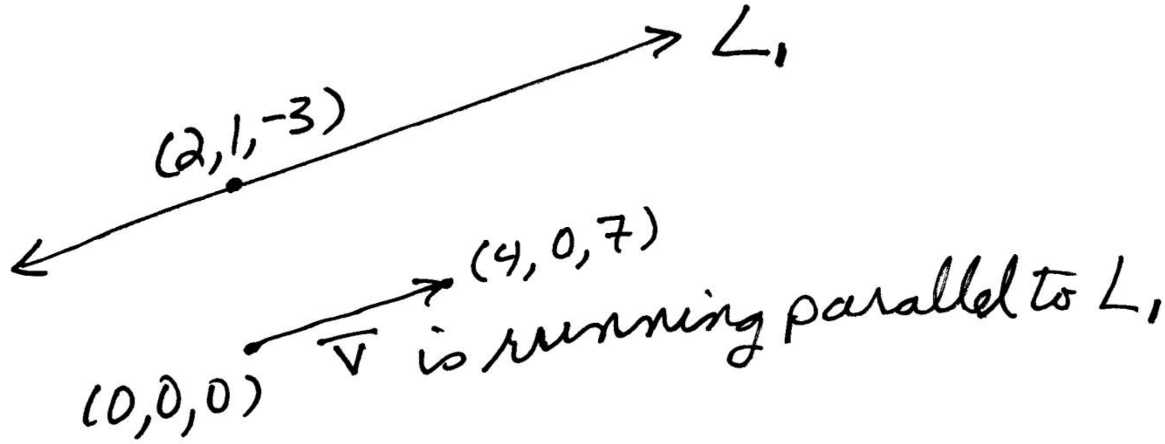
The same "t" will <u>not</u> generate $(1, 2, 3)$.

Note: Just as, given a point $\bar{p}$ and parallel vector $\bar{v}$, we can state either the vector form or parametric equations of a line; if given the equation of a line, we should be able to read off $\bar{p}$ and $\bar{v}$.

eg. Line "L_1" has the equation

$$(x, y, z) = (2, 1, -3) + t(4, 0, 7)$$

Then $\bar{p} = (2, 1, -3)$ is a point on the line and $\bar{v} = (4, 0, 7)$ is parallel to "L_1"



eg. Are the two lines $x = 3t$, $y = 4 - t$, $z = 6 + 2t$ and $x = 2 - 6s$, $y = 1 + 2s$, $z = -4s$ parallel to each other?

Note: When talking about 2 lines in the same problem, we must use 2 different parameters.

Line these equations up in columns to read off $\overline{p}$ and $\overline{v}$.

$$\begin{aligned} x &= \;\; 3t \\ y &= 4 - t \\ z &= 6 + 2t \end{aligned} \qquad\qquad \begin{aligned} x &= 2 - 6s \\ y &= 1 + 2s \\ z &= \; -4s \end{aligned}$$

$\overline{p_1} = (0,4,6)$ $\overline{v_1} = (3,-1,2)$ $\qquad$ $\overline{p_2} = (2,1,0)$ $\overline{v_2} = (-6,2,-4)$

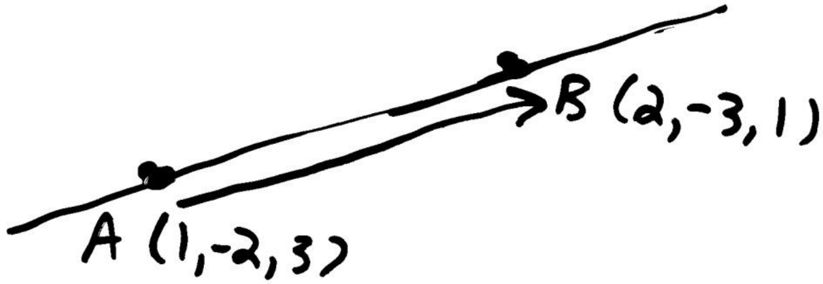
$\boxed{\text{2 Lines are parallel if their 2 vectors are parallel}}$

$\overline{v_1} = (3,-1,2)$ and $\overline{v_2} = (-6,2,-4)$

$\overline{v_2} = -2\overline{v_1}$! ($\overline{v_2}$ is a scalar multiple of $\overline{v_1}$) $\therefore \overline{v_1} \parallel \overline{v_2}$

$\boxed{\text{The 2 lines are parallel.}}$

3.



Line: need a point → use A

need a parallel vector → $\overrightarrow{AB}$

$\overrightarrow{AB} = \bar{b} - \bar{a} = (2,-3,1) - (1,-2,3)$

vector $\boxed{\overrightarrow{AB} = (1,-1,-2)}$ // vector

point $A = (1,-2,3)$ point

$$\boxed{\begin{aligned} x &= 1 + 1t \\ y &= -2 - 1t \\ z &= 3 - 2t \end{aligned}}$$

$\boxed{x = 1 + t,\ y = -2 - t,\ z = 3 - 2t}$

4. Given $x = 1+t$, $y = 2-3t$, $z = 4+2t$ and $x = 9+7s$, $y = 4+5s$, $z = 9+3s$

Sub the parametric equations for the first line in for x, y and z in the second line to see where the two lines intersect.

$$1+t = 9+7s$$
$$2-3t = 4+5s$$
$$4+2t = 9+3s$$

Collect the "s" and "t" terms on the left side of each equation and move the numbers to the right.

$$t - 7s = 8$$
$$-3t - 5s = 2$$
$$2t - 3s = 5$$

We now have three equations with two unknowns. Set up an augmented matrix and row-reduce.

$$\begin{array}{cc} t & s \end{array}$$
$$\left(\begin{array}{cc|c} 1 & -7 & 8 \\ -3 & -5 & 2 \\ 2 & -3 & 5 \end{array}\right) \begin{array}{l} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array} \quad \left(\begin{array}{cc|c} 1 & -7 & 8 \\ 0 & -26 & 26 \\ 0 & 11 & -11 \end{array}\right) R_2 \to -\tfrac{1}{26} R_2$$

$$\left(\begin{array}{cc|c} 1 & -7 & 8 \\ 0 & 1 & -1 \\ 0 & 11 & -11 \end{array}\right) \begin{array}{l} R_1 \to R_1 + 7R_2 \\ R_3 \to R_3 - 11R_2 \end{array} \quad \begin{array}{c} t \quad s \\ \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right) \end{array}$$

DONE $\boxed{t = 1 \quad s = -1}$

Sub $t=1$ into $x = 1+\underbrace{t}_{1}$, $y = 2-3\underbrace{t}_{1}$, $z = 4+2\underbrace{t}_{1}$

and $s=-1$ into $x = 9+7\underbrace{s}_{-1}$, $y = 4+5\underbrace{s}_{-1}$, $z = 9+3\underbrace{s}_{-1}$

Both lines come up with $x = 2, y = -1, z = 6$ which proves the point $(2, -1, 6)$ is on both lines.

$\boxed{(2,-1,6) \text{ is the point of intersection.}}$

5. Given: $\begin{cases} x+2y-3z=5 \\ x+3y+z=9 \end{cases}$

This is simply a system of two equations with three unknowns. Solve this system and you will discover where the two planes intersect.

Set up an augmented matrix and row-reduce.

$$\begin{array}{ccc} x & y & z \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 1 & 3 & 1 & 9 \end{array}\right) R_2 \to R_2 - R_1$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & 1 & 4 & 4 \end{array}\right) R_1 \to R_1 - 2R_2$$

$$\begin{array}{c} \qquad\quad t \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & 0 & -11 & -3 \\ 0 & 1 & 4 & 4 \end{array}\right) \text{"}z\text{" is a parameter (Let } z = t.)$$

$$\begin{cases} x = -3 + 11t \\ y = 4 - 4t \\ z = t \end{cases}$$

Note, the infinite solutions with one parameter, t, look just like the parametric equation of a line!

We have proven these two planes have a line of intersection and the equation of that line is

$x = -3 + 11t,\ y = 4 - 4t,\ z = t$.

<u>Note</u>: If you are given a system of equations with 3 unknowns "x, y, z", you have been given a system of <u>planes</u>. Solving the system determines how the planes intersect:

(i) No Solution → They don't intersect.

(ii) One solution → There is a <u>point</u> of intersection (x_0, y_0, z_0).

(iii) Infinite Solutions with <u>one</u> parameter → There is a <u>line</u> of intersection.

(iv) Infinite Solutions with <u>two</u> parameters → There is a <u>plane</u> of intersection.

Homework:

- Memorize the **Equations of Lines** on page 453.
- Study the lesson thoroughly until you can do **questions 1 to 5** on page 453 from start to finish without any assistance.
- Do all of the **Practise Problems** below (solutions are on page 463).

Practise Problems:

1. Find parametric equations of the line L passing through the points $(1, -2, 2)$ and $(3, 5, 6)$.

2. Let $P = (2, -3, 6)$ and $Q = (4, 9, 10)$ be two points.

(a) Find the vector equation of the line through P and Q.

(b) Find the parametric equations for the line through P and Q.

3. Let $P = (1, 2, 3)$ and $Q = (4, 5, 6)$ be two points.

(a) Find the vector equation of the line through P and Q.

(b) Find parametric equations for the line through P and Q.

4. Let $P = (0, 0, 0)$, $Q = (0, 2, 0)$, and $R = (0, 0, 1)$.

Find a parametric equation of the line through P, parallel to $\overrightarrow{QR}$.

1.

Let $D = (1,-2,2)$, $E = (3,5,6)$

$\overrightarrow{DE} = (3,5,6)-(1,-2,2) = (2,7,4)$ is parallel to L

Use $\bar{v} = (2,7,4)$ and point D as $\bar{p} = (1,-2,2)$

Parametric Equations are $\bar{x} = \bar{p} + t\bar{v}$ in columns

$$\begin{aligned} x &= 1 + 2t \\ y &= -2 + 7t \\ z &= 2 + 4t \end{aligned} \qquad \underbrace{\quad}_{\bar{p}\ +\ t\bar{v}}$$

ie $\boxed{x = 1+2t,\ y = -2+7t,\ z = 2+4t}$

2. (a) and (b)

2. (a) $\overrightarrow{PQ} = (4,9,10)-(2,-3,6) = (2,12,4)$

$\bar{v} = (2,12,4)$ is parallel to the line

Let $\bar{p} = (2,-3,6)$ (using point P)

Vector Form: $\bar{x} = \bar{p} + t\bar{v}$ $\boxed{(x,y,z) = (2,-3,6)+t(2,12,4)}$

(b) Parametric Equations:

$$\begin{aligned} x &= 2 + 2t \\ y &= -3 + 12t \\ z &= 6 + 4t \end{aligned} \qquad \bar{p}\ +t\bar{v}$$

$\boxed{x = 2+2t,\ y = -3+12t,\ z = 6+4t}$

3. (a) and (b)

$\overrightarrow{PQ} = (4,5,6)-(1,2,3) = (3,3,3)$

Any multiple of $(3,3,3)$ is parallel to the line.

Divide by 3 to get a tidier multiple:

Let $\bar{v} = (1,1,1)$ and $\bar{p} = (1,2,3)$ (Using point P)

Vector equation: $\bar{x} = \bar{p} + t\bar{v} \rightarrow \boxed{(x,y,z) = (1,2,3)+t(1,1,1)}$

(b) Parametric equations:

$$\begin{aligned} x &= 1 + 1t \\ y &= 2 + 1t \\ z &= 3 + 1t \end{aligned} \qquad \bar{p}\ +t\bar{v}$$

$\boxed{x = 1+t,\ y = 2+t,\ z = 3+t}$

4.

$\overrightarrow{QR} = (0,0,1)-(0,2,0) = (0,-2,1)$

Use $\bar{v} = (0,-2,1)$ and $\bar{p} = (0,0,0)$ (point P)

$$\begin{aligned} x &= 0 + 0t \\ y &= 0 - 2t \\ z &= 0 + 1t \end{aligned} \qquad \bar{p} + t\bar{v}$$

Parametric Equations are:

$\boxed{x = 0,\ y = -2t,\ z = t}$