DEPARTMENT & COURSE NO: MATH 1700-D01 **TIME:** 2 hours
EXAMINATION: Calculus II **EXAMINER:** M. Virgilio

1.

Evaluate $\lim\limits_{x\to-\infty}\left(1-\dfrac{3}{x}\right)^{2x}$ **8 Marks**

$$= \boxed{e^{-6}}$$

$$\lim_{x\to-\infty} \ln\left(1-\frac{3}{x}\right)^{2x}$$

$$= \lim_{x\to-\infty} 2x \ln\left(1-\frac{3}{x}\right)$$

$$= \lim_{x\to-\infty} \frac{2\ln(1-3x^{-1})}{x^{-1}}$$

$$\overset{[H]}{=} \lim_{x\to-\infty} \frac{2\left(\frac{3\cancel{x^{-2}}}{1-3x^{-1}}\right)}{\cancel{-x^{-2}}}$$

$$= \lim_{x\to-\infty} \frac{-6}{1-3/x}$$

$$= \frac{-6}{1-0}$$

$$= -6$$

2.
Show that the improper integral $\displaystyle\int_0^2 \frac{dx}{(1-x)^2}$ converges and find its value, or show that it diverges. **8 Marks**

Improper at $x=1$

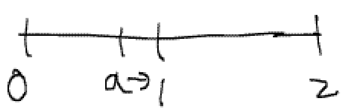
$$\int_0^2 \frac{dx}{(1-x)^2} = \int_0^1 \frac{dx}{(1-x)^2} + \int_1^2 \frac{dx}{(1-x)^2}$$

$$\int \frac{dx}{(1-x)^2} = \int (1-x)^{-2}\,dx \qquad \begin{aligned} u &= 1-x \\ du &= -dx \\ -du &= dx \end{aligned}$$

$$= -\int u^{-2}\,du$$

$$= \quad u^{-1} = \frac{1}{u} = \frac{1}{1-x}$$

$$\int \frac{dx}{(1-x)^2} = \frac{1}{1-x}$$



$$\int_0^a \frac{dx}{(1-x)^2} = \frac{1}{1-x}\Bigg|_0^a = \frac{1}{1-a} - 1$$

$$\int_0^1 \frac{dx}{(1-x)^2} = \lim_{a\to 1^-}\Big(\frac{1}{\underbrace{1-a}_{0^+}} - 1\Big) = \infty - 1 = \infty$$

$\displaystyle\int_0^1 \frac{dx}{(1-x)^2}$ diverges to ∞

$\therefore \displaystyle\int_0^2 \frac{dx}{(1-x)^2}$ diverges

3.
Solve the following integrals:

(a) $\int x^2(3-x)^5\,dx$ **8 Marks**

$$\int x^2(3-x)^5\,dx$$

$$= \int (9-6u+u^2)\,u^5 \cdot -du$$

$$= \int (-9u^5+6u^6-u^7)\,du$$

$$= \frac{-9}{6}u^6 + \frac{6}{7}u^7 - \frac{1}{8}u^8$$

$$= \boxed{-\frac{3}{2}(3-x)^6 + \frac{6}{7}(3-x)^7 - \frac{1}{8}(3-x)^8 + C}$$

$u = 3-x$

$du = -dx$

$-du = dx$

$x = 3-u$

$x^2 = (3-u)^2$

$= (3-u)(3-u)$

$= 9-6u+u^2$

(b) $\int_{1/3}^{e} 3(\ln 3x)^2\, dx$ **10 Marks**

$\int 3(\ln 3x)^2\,dx$ Int. by Parts

$u = (\ln 3x)^2 \quad du = 2(\ln 3x)\frac{3}{3x}dx$

$= \frac{2(\ln 3x)}{x}dx$

$dv = 3dx \quad v = 3x$

$$\int 3(\ln 3x)^2 dx = 3x(\ln 3x)^2 - \underbrace{\int 6(\ln 3x)\,dx}_{\text{Int by Parts}}$$

$u = \ln 3x \quad du = \frac{\cancel{3}}{\cancel{3}x}dx$

$dv = 6dx \quad v = 6x$

$$\int 3(\ln 3x)^2 dx = 3x(\ln 3x)^2 - \left[6x\ln 3x - \int 6\,dx\right]$$
$$= 3x(\ln 3x)^2 - 6x\ln 3x + 6x$$

$$\int_{1/3}^{e} 3(\ln 3x)^2 dx = \left[3x(\ln 3x)^2 - 6x\ln 3x + 6x\right]\Big|_{1/3}^{e}$$

$$= 3e(\ln 3e)^2 - 6e\ln 3e + 6e - [\underset{\ln 1}{0} - \underset{\ln 1}{0} + 2]$$

$$= \boxed{3e[\ln(3e)]^2 - 6e\ln(3e) + 6e - 2}$$

(c) $\int \cos^3 x \sin^4 x \, dx$ **8 Marks**

$$\int \cos^3 x \sin^4 x \, dx$$

$$= \int \cos^2 x \sin^4 x \cos x \, dx$$

$$= \int (1-\sin^2 x) \sin^4 x \cos x \, dx$$

$$= \int (1-u^2) u^4 \, du$$

$$= \int (u^4 - u^6) \, du$$

$$= \frac{1}{5}u^5 - \frac{1}{7}u^7$$

$$= \boxed{\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C}$$

$u = \sin x$

$du = \cos x \, dx$

(d) $\displaystyle\int \frac{x^2}{(1-9x^2)^{3/2}}\,dx.$ **12 Marks**

$$\int \frac{x^2}{(1-9x^2)^{3/2}}\,dx$$

$$= \int \frac{\frac{1}{9}\sin^2\theta \cdot \frac{1}{3}\cancel{\cos\theta}\,d\theta}{\cancel{\cos^3\theta}\ \cos^2\theta}$$

$$= \frac{1}{27}\int \tan^2\theta\,d\theta$$

$$= \frac{1}{27}\int (\sec^2\theta - 1)\,d\theta$$

$$= \frac{1}{27}\left[\tan\theta - \theta\right]$$

$$= \frac{1}{27}\left[\frac{3x}{\sqrt{1-9x^2}} - \sin^{-1}(3x)\right]$$

$$= \boxed{\frac{x}{9\sqrt{1-9x^2}} - \frac{\sin^{-1}(3x)}{27} + C}$$

$1-9x^2$

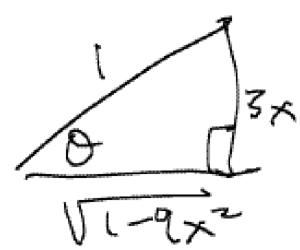
$1-\sin^2\theta$

$9x^2 = \sin^2\theta$

$x^2 = \frac{1}{9}\sin^2\theta$

$x = \frac{1}{3}\sin\theta$

$dx = \frac{1}{3}\cos\theta\,d\theta$



$\sin\theta = \frac{3x}{1}$

$$(1-9x^2)^{3/2} = (1-\sin^2\theta)^{3/2}$$
$$= (\cos^2\theta)^{3/2}$$
$$= \left(\sqrt{\cos^2\theta}\right)^3$$
$$= \cos^3\theta$$

(e) $\displaystyle\int \frac{2x^3 + 10x}{(x^2+1)^2}\, dx.$ **12 Marks**

$$= \int \frac{2x^3}{(x^2+1)^2}dx + \int \frac{10x}{(x^2+1)^2}dx$$

$u = x^2+1$
$du = 2x\,dx$
$\frac{du}{2} = x\,dx$
$\to x^2 = u-1$

$$= \int 2x^2(x^2+1)^{-2}\cdot x\,dx + \int 10(x^2+1)^{-2}\cdot x\,dx$$

$$= \int \cancel{2}(u-1)u^{-2}\frac{du}{\cancel{2}} + \int \overset{5}{\cancel{10}}\,u^{-2}\frac{du}{\cancel{2}}$$

$$= \int (u^{-1} - u^{-2})du + \int 5u^{-2}du$$

$$= \int \left(\frac{1}{u} - u^{-2}\right)du + 5\int u^{-2}du$$

$$= \ln|u| + u^{-1} - 5u^{-1}$$

$$= \ln|u| - 4u^{-1} = \ln|u| - \frac{4}{u}$$

$$= \boxed{\ln|x^2+1| - \frac{4}{x^2+1} + C}$$

4.
Find the length of the curve defined by $(y+1)^2 = 4x^3$ between the points $(0,-1)$ and $(1,1)$. **7 Marks**

$$(y+1)^2 = 4x^3$$

$$y+1 = \sqrt{4x^3} = 2x^{3/2}$$

$$\boxed{y = 2x^{3/2} - 1}$$

$$\frac{dy}{dx} = 3x^{1/2} = 3\sqrt{x}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + (3\sqrt{x})^2 = 1 + 9x$$

L from $(0,-1)$ to $(1,1)$ $\rightarrow$ $x=0$ to $x=1$

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1+9x}\, dx$$

$$\int (1+9x)^{1/2} dx \quad \rightarrow \quad u = 1+9x$$

$$du = 9dx \rightarrow \frac{du}{9} = dx$$

$$= \frac{1}{9}\int u^{1/2} du = \frac{1}{9}\cdot\frac{2}{3}u^{3/2} = \frac{2}{27}(1+9x)^{3/2}$$

$$L = \int_0^1 \sqrt{1+9x}\, dx = \frac{2}{27}(1+9x)^{3/2}\Big|_0^1$$

$$= \frac{2}{27}(10)^{3/2} - \frac{2}{27}(1)^{3/2} = \boxed{\frac{2}{27}(10)^{3/2} - \frac{2}{27}}$$

5.

For each of the following region, give a rough sketch of the region described, and set up a definite integral (or combination of definite integrals) to find the area of the region. **DO NOT SOLVE THE DEFINITE INTEGRALS.**

(a) The region between the curves $y = |4x - 1|$ and $y = x^2 - 5$, $x = 0$, $x = 4$.

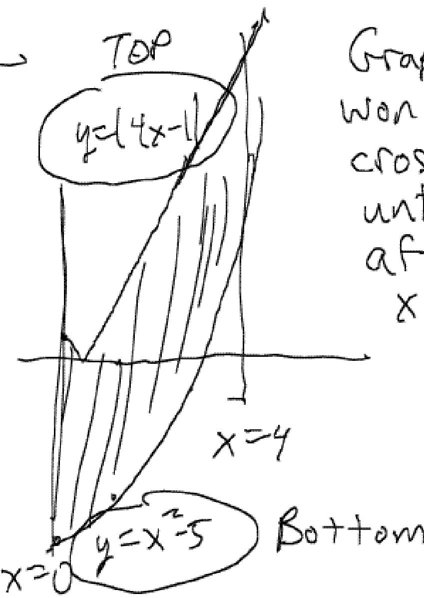
6 Marks

P of I Set $y = |4x-1|$ equal to $y = x^2-5$ between $x=0$ and $x=4$

Too hard to find points of intersection

Sketch first

x	$y=\|4x-1\|$	$y=x^2-5$
0	$\|-1\|=1$	-5
1	$\|3\|=3$	-4
2	$\|7\|=7$	-1
3	$\|11\|=11$	4
4	$\|15\|=15$	11



Graphs won't cross until after $x=4$

$$\text{Area} = \int_a^b (y_1 - y_2)\,dx$$

$$\boxed{\text{Area} = \int_0^4 \left[|4x-1| - (x^2-5)\right] dx}$$

(b) The region (described in polar coordinates) inside the circle $r = 6\cos\theta$ and outside the cardioid $r = 2(1 + \cos\theta)$. **6 Marks**

P of I: Set $r = 6\cos\theta$ equal to $r = 2(1+\cos\theta)$

$6\cos\theta = 2(1+\cos\theta)$

$6\cos\theta = 2 + 2\cos\theta$

$4\cos\theta = 2$

$\cos\theta = \frac{2}{4} = \frac{1}{2}$

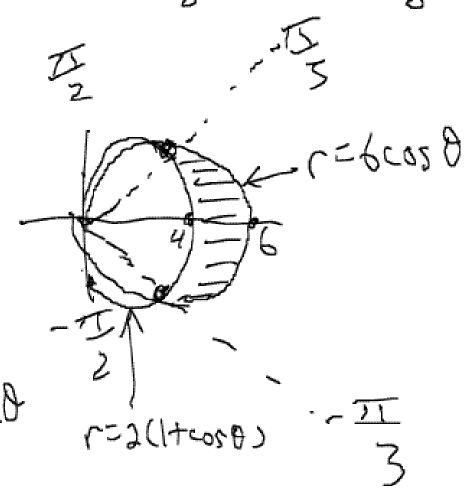
$\theta = \frac{\pi}{3}$ and $\frac{-\pi}{3}$

NASTY!

Since $r=6\cos\theta$ would be fully drawn in $\theta=0$ to $\theta=\pi$, Problem is only one P of I $\theta=\frac{\pi}{3}$

Instead draw both graphs from $\theta=-\frac{\pi}{2}$ to $\theta=\frac{\pi}{2}$ to generate 2 P of I en route $\rightarrow$ $\theta=-\frac{\pi}{3}$ and $\theta=\frac{\pi}{3}$

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$r=6\cos\theta$	0	3	6	3	0
$r=2(1+\cos\theta)$	2	3	4	3	2



$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (6\cos\theta)^2\,d\theta - \frac{1}{2}\int_{-\pi/3}^{\pi/3} [2(1+\cos\theta)]^2\,d\theta$$

$$\boxed{= \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left[(6\cos\theta)^2 - [2(1+\cos\theta)]^2\right] d\theta}$$

6.
Set up **BUT DO NOT SOLVE** the integrals which give the volume of solid obtained by rotating the region bounded by $y = 4 - x^2$, the x-axis and the y-axis:

(a) about the x-axis. **5 Marks**

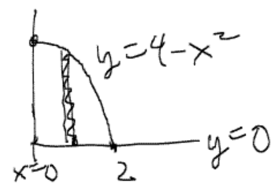
(b) about the y-axis. **5 Marks**

(c) about the line $x = -2$. **5 Marks**

$y=4-x^2$, $y=0$ (x-axis), $x=0$ (y-axis) → use dx

P of I → $4-x^2=0$

$x=2$ or $\cancel{x=-2}$ irrelevant



(a) dx around x-axis → Washer

$$V=\pi\int_a^b (y_1^2 - y_2^2)\,dx \qquad y_1 = 4-x^2,\quad y_2 = 0$$

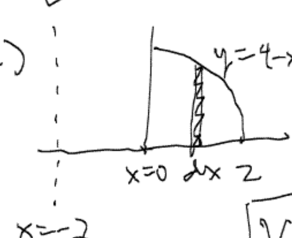
$$\boxed{V=\pi\int_0^2 (4-x^2)^2\,dx}$$

(b) dx around y-axis → Cylindrical Shell

$$V=2\pi\int_a^b x(y_1-y_2)\,dx$$

$$\boxed{V=2\pi\int_0^2 x(4-x^2)\,dx}$$

(c)



dx around $x=-2$ → Cylindrical Shell

$x+2=0$

$$V=2\pi\int_0^2 (x+2)(y_1-y_2)\,dx$$

$$\boxed{V=2\pi\int_0^2 (x+2)(4-x^2)\,dx}$$