DEPARTMENT & COURSE NO: MATH 1700-D01 **TIME:** 2 hours
EXAMINATION: Calculus II **EXAMINER:** M. Virgilio

1.
Evaluate $\lim_{x\to 0^+} (1+3x)^{\csc x}$ **8 Marks**

$$= \boxed{e^3}$$

$$\lim_{x\to 0^+} \ln\left((1+3x)^{\csc x}\right)$$

$$= \lim_{x\to 0^+} \csc x \cdot \ln(1+3x)$$

$$= \lim_{x\to 0^+} \frac{\ln(1+3x)}{\sin x}$$

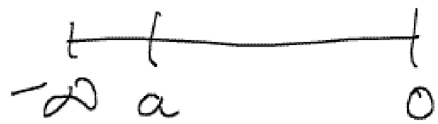
$$\overset{[H]}{=} \lim_{x\to 0^+} \frac{\frac{3}{1+3x}}{\cos x}$$

$$= \frac{\frac{3}{1}}{\cos 0} = \frac{3}{1} = \boxed{3}$$

2.

Determine whether the improper integral $\int_{-\infty}^{0} \frac{dx}{(x-8)^{2/3}}$ converges or diverges. If it converges, find its value. **8 Marks**

Improper at $-\infty$



$$\int \frac{dx}{(x-8)^{2/3}} = \int (x-8)^{-2/3}\,dx \qquad u = x-8, \quad du = dx$$

$$= \int u^{-2/3}\,du = 3u^{1/3} = 3(x-8)^{1/3}$$

$$\int_a^0 \frac{dx}{(x-8)^{2/3}} = 3(x-8)^{1/3}\Big|_a^0 = 3(-8)^{1/3} - 3(a-8)^{1/3}$$

$$3\sqrt[3]{-8} = 3(-2) = -6$$

$$= -6 - 3(a-8)^{1/3}$$

$$\int_{-\infty}^0 \frac{dx}{(x-8)^{2/3}} = \lim_{a \to -\infty}\left[6 - 3(a-8)^{1/3}\right] = 6 + \infty$$

$$6 - 3\sqrt[3]{-\infty} =$$

$$= \infty$$

$$\boxed{\int_{-\infty}^0 \frac{dx}{(x-8)^{2/3}} \text{ diverges to } \infty}$$

3.

Solve the following integrals:

(a) $\int_0^4 x^3(x^2+1)^{-1/2}\,dx$ **8 Marks**

$$\int x^3(x^2+1)^{-1/2}\,dx$$

$$u = x^2+1 \rightarrow x^2 = u-1$$

$$du = 2x\,dx \rightarrow \frac{du}{2} = x\,dx$$

$$= \int x^2(x^2+1)^{-1/2}\,x\,dx$$

$$= \int (u-1)\,u^{-1/2}\,\frac{du}{2}$$

$$= \frac{1}{2}\int \left(u^{1/2} - u^{-1/2}\right)du$$

$$= \frac{1}{2}\left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right]$$

$$= \frac{1}{3}u^{3/2} - u^{1/2}$$

$$= \frac{1}{3}(x^2+1)^{3/2} - (x^2+1)^{1/2}$$

$$= \left[\frac{1}{3}(x^2+1)^{3/2} - (x^2+1)^{1/2}\right]\Bigg|_0^4$$

$$= \frac{1}{3}(17)^{3/2} - (17)^{1/2} - \left[\frac{1}{3}(1)^{3/2} - (1)^{1/2}\right]$$

$$\left[\frac{1}{3} - 1\right]$$

$$= \boxed{\frac{(17)^{3/2}}{3} - (17)^{1/2} + \frac{2}{3}}$$

(b) $\int x^2 e^{3x}\, dx$ **10 Marks**

Int. by Parts $u = x^2 \rightarrow du = 2x\,dx$

$dv = e^{3x}dx \rightarrow v = \frac{1}{3}e^{3x}$

$$\int x^2 e^{3x} dx = \frac{x^2}{3}e^{3x} - \frac{2}{3}\int x e^{3x} dx$$

$u = x \longrightarrow du = dx$

$dv = e^{3x}dx \rightarrow v = \frac{1}{3}e^{3x}$

$$= \frac{x^2}{3}e^{3x} - \frac{2}{3}\left[\frac{x}{3}e^{3x} - \frac{1}{3}\int e^{3x} dx\right]$$

$$\boxed{\int x^2 e^{3x} dx = \frac{x^2}{3}e^{3x} - \frac{2x}{9}e^{3x} + \frac{2}{27}e^{3x} + C}$$

(c) $\int \sin^5 x \cos^3 x\, dx$ **8 Marks**

$u = \sin x \rightarrow du = \cos x\, dx$

$\int \sin^5 x \cos^3 x\, dx = \int \sin^5 x \cos^2 x \cos x\, dx$

$= \int \sin^5 x (1 - \sin^2 x) \cos x\, dx$

$= \int u^5 (1 - u^2)\, du$

$= \int (u^5 - u^7)\, du = \frac{1}{6} u^6 - \frac{1}{8} u^8$

$\int \sin^5 x \cos^3 x\, dx = \boxed{\frac{1}{6} \sin^6 x - \frac{1}{8} \sin^8 x + C}$

(d) $\displaystyle\int \frac{1}{x^3\sqrt{x^2-25}}\,dx.$ **12 Marks**

x^2-25

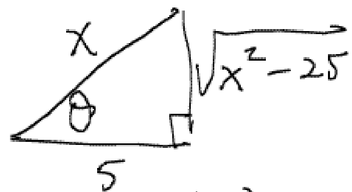
$\sec^2\theta - 1$

$x^2 = 25\sec^2\theta$

$x = 5\sec\theta$

$\sec\theta = \frac{x}{5} = \frac{H}{A}$

$dx = 5\sec\theta\tan\theta\,d\theta$



$x^3 = (5\sec\theta)^3$

$x^3 = 125\sec^3\theta$

$\theta = \sec^{-1}\left(\frac{x}{5}\right)$

$\sin\theta = \frac{O}{H} = \frac{\sqrt{x^2-25}}{x}$

$\cos\theta = \frac{A}{H} = \frac{5}{x}$

$$\sqrt{x^2-25} = \sqrt{25\sec^2\theta - 25} = \sqrt{25(\sec^2\theta-1)} = \sqrt{25\tan^2\theta} = 5\tan\theta$$

$$\int \frac{dx}{x^3\sqrt{x^2-25}} = \int \frac{\cancel{5}\cancel{\sec\theta}\cancel{\tan\theta}\,d\theta}{125\underbrace{\cancel{\sec^3\theta}}_{\sec^2\theta}\cdot\cancel{5}\cancel{\tan\theta}} \rightarrow \frac{1}{\sec^2\theta} = \cos^2\theta$$

$$= \frac{1}{125}\int \cos^2\theta\,d\theta = \frac{1}{125}\cdot\frac{1}{2}\int(1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{250}\left[\theta + \frac{1}{2}\underbrace{\sin 2\theta}_{2\sin\theta\cos\theta}\right] = \frac{1}{250}\left[\theta + \sin\theta\cos\theta\right]$$

$$= \frac{1}{250}\left[\sec^{-1}\left(\frac{x}{5}\right) + \frac{\sqrt{x^2-25}}{x}\cdot\frac{5}{x}\right]$$

$$= \boxed{\frac{1}{250}\sec^{-1}\frac{x}{5} + \frac{1}{50}\frac{\sqrt{x^2-25}}{x^2} + C}$$

(e) $\displaystyle\int \frac{2x^2 - 25x - 33}{(x+1)^2(x-5)}\,dx.$ **12 Marks**

Partial Fractions

$$\frac{2x^2-25x-33}{(x+1)^2(x-5)} = \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{C}{x-5}$$

$$A = \left.\frac{2x^2-25x-33}{x-5}\right|_{x=-1} = \frac{-6}{-6} \longrightarrow A=1$$

$$C = \left.\frac{2x^2-25x-33}{(x+1)^2}\right|_{x=5} = \frac{-108}{36} \longrightarrow C=-3$$

$$2x^2-25x-33 = \overset{1}{\cancel{A}}(x-5) + B(x+1)(x-5) + \overset{-3}{\cancel{C}}(x+1)^2$$

Sub in $x=0$: $-33 = -5 - 5B - 3$

$-25 = -5B \longrightarrow B=5$

$$\frac{2x^2-25x-33}{(x+1)^2(x-5)} = \frac{1}{(x+1)^2} + \frac{5}{x+1} - \frac{3}{x-5}$$

$$\int \frac{2x^2-25x+33}{(x+1)^2(x-5)}\,dx = \underbrace{\int (x+1)^{-2}\,dx}_{\int u^{-2}du = -u^{-1} = \frac{-1}{u}} + 5\int \frac{dx}{x+1} - 3\int \frac{dx}{x-5}$$

$$= \boxed{\frac{-1}{x+1} + 5\ln|x+1| - 3\ln|x-5| + C}$$

4.
Find the length of the curve defined by $(y+1)^2 = (x-4)^3$ between the points $(5,0)$ and $(8,7)$. **7 Marks**

$(y+1)^2 = (x-4)^3$ from $(5,0)$ to $(8,7)$

$y+1 = (x-4)^{3/2}$

$\boxed{y = (x-4)^{3/2} - 1} \rightarrow$ Use dx from $x=5$ to $x=8$

$\frac{dy}{dx} = \frac{3}{2}(x-4)^{1/2}$

$\left(\frac{dy}{dx}\right)^2 = \frac{9}{4}(x-4) = \frac{9}{4}x - 9$

$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{9}{4}x - 9 = \frac{9}{4}x - 8$

$\int \sqrt{\frac{9}{4}x - 8}\, dx = \int \left(\frac{9}{4}x - 8\right)^{1/2} dx$

$u = \frac{9}{4}x - 8 \rightarrow du = \frac{9}{4}dx \rightarrow dx = \frac{4}{9}du$

$= \frac{4}{9}\int u^{1/2} du = \frac{4}{9}\cdot\frac{2}{3}u^{3/2} = \frac{8}{27}\left(\frac{9}{4}x - 8\right)^{3/2}$

$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_5^8 \sqrt{\frac{9}{4}x - 8}\, dx$

$= \frac{8}{27}\left(\frac{9}{4}x - 8\right)^{3/2}\Big|_5^8 = \frac{8}{27}(10)^{3/2} - \frac{8}{27}\Big(\underbrace{\frac{45}{4} - 8}_{\frac{45-32}{4}}\Big)^{3/2}$

$= \boxed{\frac{8}{27}(10)^{3/2} - \frac{8}{27}\left(\frac{13}{4}\right)^{3/2}}$

5.

For each of the following region, give a rough sketch of the region described, and set up a definite integral (or combination of definite integrals) to find the area of the region. **DO NOT SOLVE THE DEFINITE INTEGRALS.**

(a) The region between the curves $y = x + 6$, $y = x^3$, and $2y + x = 0$.

6 Marks

Find where the pairs of graphs intersect

$y = x + 6$ (1), $y = x^3$ (2), $2y + x = 0 \rightarrow y = -\frac{1}{2}x$ (3)

(1) = (2) $\quad x + 6 = x^3 \rightarrow x = 2$

$8 = 8$ ✓

(1) = (3) $\quad x + 6 = -\frac{1}{2}x \rightarrow$ multiply by 2

$2x + 12 = -x$

$3x = -12$

$x = -4$

(2) = (3) $\quad x^3 = -\frac{1}{2}x \rightarrow$ multiply by 2

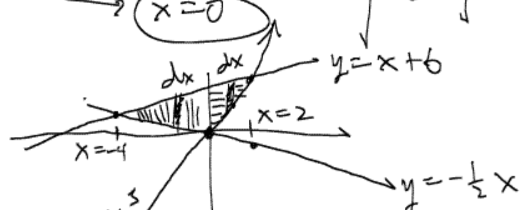
$2x^3 = -x$

$2x^3 + x = 0$

$x\underbrace{(2x^2+1)}_{\text{never } = 0} = 0$

$\rightarrow x = 0$

x	$y=x+6$	$y=x^3$	$y=-\frac{1}{2}x$
-4	2	-64	2
0	6	0	0
2	8	8	-1



$$\text{Area} = \int_a^b (y_1 - y_2)\,dx$$

A_1 is from $x = -4$ to $x = 0$

$y_1 = x + 6$ above $y_2 = -\frac{1}{2}x$

$x = 0$

A_2 is from $x = 0$ to $x = 2$

$y_1 = x + 6$ above $y_2 = x^3$

$$\boxed{\text{Area} = \int_{-4}^{0} \left(x + 6 + \tfrac{1}{2}x\right)dx + \int_{0}^{2} \left(x + 6 - x^3\right)dx}$$

(b) The region (described in polar coordinates) inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 3\cos\theta$. **6 Marks**

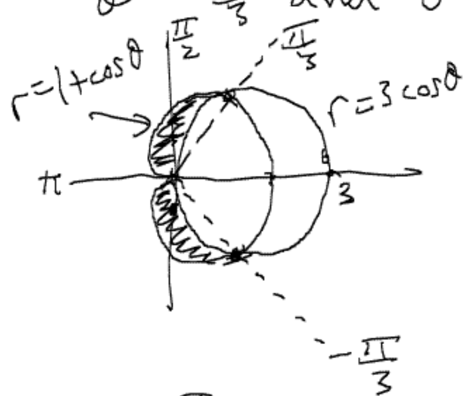
Pof I $\quad r = 1 + \cos\theta$ and $\underbrace{r = 3\cos\theta}_{\text{Full circle from } -\frac{\pi}{2} \text{ to } \frac{\pi}{2} \text{ or } 0 \text{ to } \pi}$

$1 + \cos\theta = 3\cos\theta$

$1 = 2\cos\theta$

$\cos\theta = \frac{1}{2}$

$\theta = -\frac{\pi}{3}$ and $\theta = \frac{\pi}{3}$



θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
$r = 1+\cos\theta$	1	$\frac{3}{2}$	2	$\frac{3}{2}$	1	

Shaded region is inside the cardioid but outside the circle

From $\theta = \frac{\pi}{3}$ to $\frac{\pi}{2}$ $\rightarrow$ Area of cardioid − Area of circle
From $\theta = \frac{\pi}{2}$ to π $\rightarrow$ Area of cardioid only

Total area is Double since symmetrical region below the x-axis.

$$\text{Area} = \frac{1}{2}\int_a^b r^2\, d\theta$$

$$\text{Area} = 2\left[\frac{1}{2}\int_{\pi/3}^{\pi/2} (1+\cos\theta)^2 - (3\cos\theta)^2\, d\theta\right]$$

$$+ 2\left[\frac{1}{2}\int_{\pi/2}^{\pi} (1+\cos\theta)^2\, d\theta\right]$$

$$\text{Area} = \int_{\pi/3}^{\pi/2}\left[(1+\cos\theta)^2 - (3\cos\theta)^2\right] d\theta + \int_{\pi/2}^{\pi} (1+\cos\theta)^2\, d\theta$$

6.

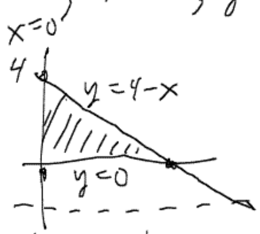
Set up **BUT DO NOT SOLVE** the integrals which give the volume of solid obtained by rotating the region bounded by $y = 4 - x$, $x = 0$, and $y = 0$ with $0 \le x \le 4$:

(a) about the x-axis. **5 Marks**

(b) about the y-axis. **5 Marks**

(c) about the line $y = -1$. **5 Marks**

$y = 4 - x$, $x = 0$, $y = 0$ $\quad 0 \le x \le 4$



Use dx from $x = 0$ to $x = 4$

$y = -1 \to y + 1 = 0$

(a) dx about the x-axis $\to$ Washer

$$V = \pi \int_a^b (y_1^2 - y_2^2)\,dx$$

$$\boxed{V = \pi \int_0^4 (4-x)^2\,dx}$$

(b) dx about the y-axis $\to$ Cyl. Shell

$$V = 2\pi \int_a^b x(y_1 - y_2)\,dx$$

$$\boxed{V = 2\pi \int_0^4 x(4-x)\,dx}$$

(c) dx about $\underbrace{y = -1}_{\text{pseudo x-axis}}$ $\to$ Washer

pseudo x-axis $\to$ Don't say this on the exam

$$V = \pi \int_a^b (\underset{\substack{\downarrow \\ 4-x}}{y_1} + 1)^2 - (\underset{\substack{\downarrow \\ 0}}{y_2} + 1)^2\,dx$$

$$\boxed{V = \pi \int_0^4 [(5-x)^2 - 1]\,dx}$$